Home work 2 Model Answers Stat 310

1. We have the following information:
   5% of the population is gay.
   Gaydar correctly identifies a gay person 56% of the time
   Gaydar incorrectly identifies a straight person as gay 30% of the time

a. (3pts) What is the probability that a person is gay given that your gaydar says they are? Hint: use Bayes Rule

<u>ANSWER</u>: According to Bayes Rule

$$P(\text{Gay} \mid \text{Gaydar says Gay}) = \frac{P(\text{Gay})P(\text{Gaydar says gay} \mid \text{Gay})}{P(\text{Gay})P(\text{Gaydar says gay} \mid \text{Gay}) + P(\text{Not Gay})P(\text{Gaydar says gay} \mid \text{Not Gay})} \quad \Big\} \text{ 1pt}$$

Luckily, we are given all of these probabilities above

$P(\text{Gay}) = 5\% = 0.05$

$P(\text{Gaydar says gay} \mid \text{Gay}) = 0.56$ } 1pt

$P(\text{Gaydar says gay} \mid \text{Not Gay}) = 0.30$

And we can calculate

$P(\text{Not Gay}) = 1 - P(\text{Gay}) = 1 - 0.05 = 0.95$ } 1pt

(because Not Gay and Gay are the only two possibilities)

Plugging in we have

$$P(\text{Gay} \mid \text{Gaydar says gay}) = \frac{0.05 \times 0.56}{0.05 \times 0.56 + 0.95 \times 0.30}$$

$$\boxed{= 0.0895}$$

which means that only about 9% of the people who set off our gaydar are actually gay.

1.b. (2pts) In San francisco, 15.4% of the population is gay. How does this change the probability that your gaydar is right? Use natural frequencies.

ANSWER: In San Francisco the probabilities of being gay change: $P(\text{Gay}) = 0.154$ $P(\text{Not Gay}) = 1 - 0.154 = 0.846$ } ½ point

Let's apply these probabilities to a population of 1000 people

1000 San Franciscans

$1000 \times 0.154 = 154$ Gay SFs

$1000 \times 0.846 = 846$ Straight SF

Now we apply our gaydar to all of these people and see how many we classify as gay

$154 \times 0.56 = 86.24$ Gay people we think are gay

$154 \times (1-0.56) = 67.76$ Gay people we think are straight

$846 \times 0.30 = 253.8$ Straight people we think are gay

$846 \times (1-0.30) = 592$ straight people we think are straight

½ pt for correct methods of calculating frequencies (accept any starting value)

To get the probability that a person we think is gay is actually gay, we divide the number of outcomes where a person we think is gay is actually gay by the total number of events where we think the person is gay.

$$P(\text{Gay} \mid \text{Gaydar says gay}) = \frac{\#\text{Gay \& Gaydar thinks gay}}{\#\text{Gaydar thinks gay}} = \frac{86.24}{86.24 + 253.8} = \frac{86.24}{340.04}$$

$$\boxed{= 0.254 \text{ or } 25.4\%}$$

½ point for setting up a fraction based on special cases over total cases

½ point for correctly picking which groups are the special cases and which are the total cases

1.c (3pts) What is the probability that someone is gay if <u>both</u> you and your friend think they are? Assume that the probabilitie of you and your friend thinking a person is gay are independent

1.c (continued)

ANSWER: Again we wish to know a conditional probability, which means Bayes rule might be a good way to calculate it.

$P(\text{Gay} \mid \text{Your Gaydar says gay AND friends gaydar says gay}) =$

Since this is so wordy I'm going to change my notation

Let Gay be the event that the person is Gay
Let Not Gay be the event that the person is not gay
Let You=+ be the event that your gaydar says gay
Let Friend=+ be the event that your friend's gaydar says gay

So, by Bayes rule

$$P(\text{Gay} \mid \text{You}=+ \text{ and friend}=+) = \frac{P(\text{Gay})P(\text{You}=+ \text{ and friend}=+ \mid \text{Gay})}{P(\text{Gay})P(\text{You}=+ \text{ and friend}=+ \mid \text{Gay}) + P(\text{Not Gay})P(\text{You}=+ \text{ and friend}=+ \mid \text{Not Gay})}$$ 1 pt

Since the probabilities of you and your friends gaydar going off are conditionally independent (Given in the problem), we can apply the rule of independence to simplify

$$P(\text{You}=+ \text{ and friend}=+ \mid \text{Gay}) = P(\text{You}=+ \mid \text{Gay})P(\text{friend}=+ \mid \text{Gay})$$

and

$$P(\text{You}=+ \text{ and friend}=+ \mid \text{Not Gay}) = P(\text{You}=+ \mid \text{Not Gay})P(\text{friend}=+ \mid \text{Not Gay})$$ 1 pt

Now we just need to plug in all of the known probabilities and solve. Note: the information we have suggests that everyone's gaydar works with the same probabilities. Also note: there's no reference to us being in San Francisco.

$$P(\text{Gay} \mid \text{You}=+ \text{ and friend}=+) = \frac{0.05 \times 0.56 \times 0.56}{0.05 \times 0.56 \times 0.56 + 0.95 \times 0.30 \times 0.30}$$ 1 pt

$$\boxed{= 0.155}$$

1.d (2pts) Is the assumption of independence in the previous question reasonable? why/why not?

Accept any well thought out answers that identifies a plausible reason for dependence.

No = 1pt

reason = 1pt

Example: The assumption of independence is unreasonable because the fact of friendship suggests that you and your friend have similar worldviews/ thought processes/ experiences. If one of you perceives that a person is gay, its more likely that the other one will too. This tendency to agree is a basis of friendship.

or

The assumption of <u>conditional</u> independence is unrealistic. If it were true, it would imply

$$P(\text{You}=+ \text{ and Friend}=+ | \text{Gay}) = P(\text{You}=+ | \text{Gay}) P(\text{Friend}=+ | \text{Gay})$$
$$= 0.56 \times 0.56$$
$$= 0.3136$$

But from Bayes rule

$$P(\text{You}=+ \text{ and Friend}=+ | \text{Gay}) = \frac{P(\text{Gay}) P(\text{Gay} | \text{You}=+ \text{ and friend}=+)}{P(\text{Gay}) P(\text{Gay} | \text{You}=+ \text{ and friend}=+) + P(\text{Not Gay}) P(\text{Not Gay} | \text{You}=+ \text{ and friend}}$$

$$= \frac{P(\text{Gay}) P(\text{Gay} | \text{You}=+ \text{ and friend}=+)}{P(\text{Gay}) P(\text{Gay} | \text{You}=+ \text{ and friend}=+) + P(\text{Not Gay})(1 - P(\text{Gay} | \text{You}=+ \text{ and friend}=+))}$$

If we again assume conditional independence, we can use the results of part c $(P(\text{Gay} | \text{You}=+ \text{ and friend}=+) = 0.155)$ to have

1.d. continued

$$= \frac{0.05 \times 0.155}{0.05 \times 0.155 + 0.95(1-0.155)}$$

$$= 0.00956 \neq 0.3136$$

Hence our assumption of conditional independence leads to a contradiction and can not be true.

2. Check the following two statements. Are they correct? Show your reasoning.

a. (2pts) $P(A^c|B) + P(A|B) = 1$

<u>ANSWER</u>: Yes it is correct ← 1pt

By the definition of conditional probability (applied twice in a row)

$$P(A^c|B) + P(A|B) = \frac{P(A^c \cap B)}{P(B)} + \frac{P(A \cap B)}{P(B)}$$

$$= \frac{P(A^c)P(B|A^c)}{P(B)} + \frac{P(A)P(B|A)}{P(B)}$$

$$= \frac{P(A^c)P(B|A^c) + P(A)P(B|A)}{P(B)}$$

Since $A^c \cap A = \emptyset$ and $A \cup A^c = S$, we can apply the law of total probability to the top to get

$$= \frac{P(B)}{P(B)}$$

$$= 1$$

1 pt
Any clear proof (verbal or mathematical, or pictoral)

(Alternatively, they can shorten the proof with property 2 of conditional expectations on pg. 72: $P(E|A) = 1 - P(E^c|A)$)

2.b. (2pts) $P(A|B^c) + P(A|B) = 1$

Not true

1pt

Consider the situation where A is independent of B and $P(A) \neq \frac{1}{2}$. For example, let A be the event of rolling a 6 on a fair die (i.e. singular of dice). Let B be the event of getting heads on a fair coin flip. Then

$$P(A|B^c) + P(A|B) = P(A) + P(A) \quad \text{(def. of independence)}$$
$$= \frac{1}{6} + \frac{1}{6}$$
$$= \frac{1}{3}$$
$$\neq 1$$

1pt for any counter example or clear logic

3. A, B, and C are mutually independent. Prove:

a. (3pts) A^c and B^c are independent

$$P(A^c | B^c) = 1 - P(A|B^c)$$ law of total probability

$$= 1 - \frac{P(A \cap B^c)}{P(B^c)}$$ definition of conditional independence

$$= 1 - \frac{P(A)P(B^c|A)}{P(B^c)}$$ definition of conditional independence

$$= 1 - \frac{P(A)(1-P(B|A))}{P(B^c)}$$ law of total probability

$$= 1 - \frac{P(A)(1-P(B))}{P(B^c)}$$ because B is independent of A

$$= 1 - \frac{P(A)P(B^c)}{P(B^c)}$$ because $S = B^c \cup B$ and $B^c \cap B = \emptyset$ or definition of complement or law of total probability

because A & B^c are independent as shown in class

$$= 1 - P(A)$$

$$= P(A^c)$$ law of total probability

which proves A^c is independent of B^c which implies B^c is also independent of A^c

(3 pts for any valid proof)

b. (1pt) A^c and C^c are independent and B^c and C^c are independent

$P(A^c | C^c) = P(A^c)$ by above proof (change B to C throughout) 1pt

$P(C^c | B^c) = P(C^c)$ by above proof (change A to C throughout)

C. (3pts) A^c, B^c, and C^c are mutually independent

<u>ANSWER:</u> A^c, B^c, and C^c will be mutually independent only if the elements of <u>every</u> subcollection of A^c, B^c, and C^c are independent (<u>1pt for recognizing this</u>)

We've shown in parts a. and b. that $A^c \perp B^c$ $B^c \perp A^c$ and $A^c \perp C^c$. Now we just need to show that $P(A^c \cap B^c \cap C^c) = P(A^c)P(B^c)P(C^c)$

<u>STEP 1:</u> Lets show that C^c is independent of $A^c \cap B^c$

NTS: $P(C^c \mid A^c \cap B^c) = C^c$

$$P(C^c \mid A^c \cap B^c) = 1 - P(C \mid A^c \cap B^c) \quad \text{Law of total probablity}$$

$$= 1 - \frac{P(C \cap A^c \cap B^c)}{P(A^c \cap B^c)} \quad \text{Property of conditional independence}$$

$$= 1 - \frac{P(C)P(A^c \cap B^c \mid C)}{P(A^c \cap B^c)} \quad \text{Property of conditional indepance}$$

$$= 1 - \frac{P(C)(1 - P(A \cup B \mid C))}{P(A^c \cap B^c)} \quad \text{Law of total probability } (A^c \cap B^c)^c = A \cup B$$

$$= 1 - \frac{P(C)(1 - (P(A \mid C) + P(B \mid C) - P(A \cap B \mid C)))}{P(A^c \cap B^c)} \quad A \cup B = A + B - A \cap B \text{ Listed in properties of conditional independence}$$

$$= 1 - \frac{P(C)\left(1 - \left(P(A) + P(B) - \frac{P(A \cap B \cap C)}{P(C)}\right)\right)}{P(A^c \cap B^c)} \quad \text{Property of conditional ind.}$$

$$= 1 - \frac{P(C)\left[1 - \left(P(A) + P(B) - \frac{P(A)P(B)\cancel{P(C)}}{\cancel{P(C)}}\right)\right]}{P(A^c \cap B^c)} \quad A, B, \text{ and } C \text{ are mutually independent}$$

$$= 1 - \frac{P(C)[1 - (P(A) + P(B) - P(A \cap B))]}{P(A^c \cap B^c)} \quad A \perp B \rightarrow P(A \cap B) = P(A)P(B)$$

$$= 1 - \frac{P(C)[1 - P(A \cup B)]}{P(A^c \cap B^c)} \quad P(A \cup B) = P(A) + P(B) - P(A \cap B)$$

$$= 1 - \frac{P(C)P(A^c \cap B^c)}{P(A^c \cap B^c)}$$ Law of total probability, and $(A \cup B)^c = A^c \cap B^c$

$$= 1 - P(C)$$

$$= P(C^c)$$

$$\therefore C^c \perp (A^c \cap B^c)$$

This implies that

$$P(A^c \cap B^c \cap C^c) = P((A^c \cap B^c) \cap C^c)$$

$$= P(A^c \cap B^c) P(C^c)$$ by the independence we just proved

$$= P(A^c)P(B^c)P(C^c)$$ because we showed $A^c \perp B^c$ in 3.a.

Which finishes our proof that A^c, B^c, and C^c are mutually independent

2 points for any valid proof

-1 point if the proof relies on assuming conditional independence (e.g. $P(A^c \cap B^c | C) = P(A^c|C)P(B^c|C)$), since this condition was not given.