Model Answers Homework 4 Stat 310

Total Points : 23

1. For each of the following mgfs, identify the distribution of the corresponding r.v. including its parameters.

a. (1pt) $M_X(t) = 2^{(e^t-1)}$

$= e^{\ln 2\,(e^t-1)}$

ANSWER : We can recognize the m.g.f. by comparing its form to known m.g.f.'s (for example, we can compare it to the m.g.f.'s listed on the distributions handout).

distribution = poisson (or no known distribution*) $\frac{1}{2}$pt

parameters = $\lambda = \ln 2$ $\frac{1}{2}$pt

(Also accept "No known distribution" for +1)

b. (1pt) $M_Y(t) = 0.5(1+e^t)$

$= 0.5 + 0.5e^t$

$= 1 - 0.5 + 0.5e^t$

$= 1 - p + pe^t, \ p = 0.5$

distribution : Bernoulli (or Binomial if $\downarrow$) $\frac{1}{2}$ pt

parameter : $p = 0.5$ ($p=0.5$ AND $n=1$) $\frac{1}{2}$ pt

* because of initial typo on homework page

c. (1pt) $M_Z(t) = \left(\frac{1-\lambda e^t}{1-\lambda}\right)^{-\alpha}$

$$= \left(\frac{1-\lambda}{1-\lambda e^t}\right)^{\alpha}$$

$$= \left(\frac{1-p}{1-pe^t}\right)^{r} \text{ where } p=\lambda \text{ and } r=\alpha$$

distribution : Negative Binomial $\frac{1}{2}$ pt

parameters : $p=\lambda$, $r=\alpha$ $\frac{1}{2}$ pt

2. Given the pmf $f(k) = \frac{-1}{\ln(1-p)} \frac{p^k}{k}$, $k=1,2,\dots\infty$, $0<p<1$

a. (3pts) find the m.g.f.

$$M_K(t) = E(e^{tx})$$ 1 pt for $M_k(t) = E(e^{tx})$

$$= \sum_{x=1}^{\infty} e^{tx} p(x)$$ 1 pt for $E(a) = \sum_a a\,p(a)$

$$= \sum_{x=1}^{\infty} e^{tx} \frac{-1}{\ln(1-p)} \frac{p^x}{x}$$ 1 pt for $p(x) = f(x)$ from above

$$= \frac{\ln(1-pe^t)}{\ln(1-p)}$$ (using wolfram alpha)

b (5pts) find the mean and variance (by using the mgf)

Mean $= E(x)$ 1 pt for $E(x) = \mu$

$$= M_X'(0)$$ 1 pt for $E(x) = M_X'(0)$

$$= \frac{d}{dt}\left(\frac{\ln(1-pe^t)}{\ln(1-p)}\right)\Bigg|_{t=0}$$

2.b. (Cont'd)

$$= \left.\frac{pe^t}{(pe^t-1)\ln(1-p)}\right|_{t=0}$$

$$= \frac{p}{(p-1)\ln(1-p)}$$

Variance $= E(X^2) - E(X)^2$

1pt for definition of variance

Using the mgf, we have

$$E(X^2) = M_X''(0)$$

1pt for $E(X^2) = M_X''(t)$

$$= \frac{d}{dt}\left.\left(\frac{pe^t}{(pe^t-1)\ln(1-p)}\right)\right|_{t=0}$$

$$= \left.\frac{pe^t}{(pe^t-1)^2\ln(1-p)}\right|_{t=0}$$

$$= \frac{p}{(p-1)^2\ln(1-p)}$$

$\therefore$ Variance $= E(X^2) - E(X)^2$

1pt for plugging in $E(X^2)$ and $E(X)$ to find variance

$$= \frac{p}{(p-1)^2\ln(1-p)} - \left(\frac{p}{(p-1)\ln(1-p)}\right)^2$$

$$= \frac{-p(p-2)}{(p-1)^2\ln(1-p)}$$

3. Given $f(x) = \dfrac{c}{(1+x^2)^2}$ $\quad -1<x<1$

a. (2 pts) Find the value of c to make it a pdf

ANSWER

To be a pmf, f must satisfy 2 conditions:

1.) $\int_{-1}^{1} f(x)dx = 1$

+1 pt for $\int_{-1}^{1} f(x) = 1$ test

$$\Rightarrow \int_{-1}^{1} \frac{c}{(1+x^2)^2}\,dx = 1$$

$$\frac{1}{4}(2+\pi)c = 1$$

$$c = \frac{4}{2+\pi}$$

2.) $f(x) \geq 0$ for all x

+1 pt for also checking that $f(x) \geq 0$ for all x given c

$\dfrac{4}{(2+\pi)(1+x^2)^2}$ will be ≥ 0 for $-1<x<1$

b. (1 pt) Argue that the mean should be zero without performing any calculation.

ex. The mean is the point where the function would "balance" if supported it only at the mean. Since any function of x^2 is symmetric about the origin, this will necessarily be at $x=0$.

+ 1 pt for any argument based on symmetry about the origin

3c. (5 pts) Find the mean and variance without using the Mgf

1). $\mu = E(x)$

$= \int_{-1}^{1} x f(x)\,dx$

$= \int_{-1}^{1} \frac{4x}{(2+\pi)(1+x^2)^2}\,dx$

$= \frac{2}{(2+\pi)(1+x^2)}\Big|_{-1}^{1}$

$= \frac{2}{2+\pi}\left(\frac{1}{2} - \frac{1}{2}\right)$

$= 0$

+1 pt $\mu = E(x)$

+1 pt integral definition of $E(x)$

2.) $Var(x) = E[(X - E(X))^2]$ or $E(X^2) - E(X)$

$= E[(X - 0)^2]$

$= E(x^2)$

$= \int_{-1}^{1} x^2 f(x)\,dx$

$= \int_{-1}^{1} \frac{4x^2}{(2+\pi)(1+x^2)}\,dx$

$= \frac{\pi - 2}{2 + \pi} \approx 0.222$

+1 pt $Var(x) = E(x^2) - E(x)$ or $E[(x - E(x))^2]$

+1 pt for setting $E(x)$ to μ above

+1 pt $E(x^2) = \int_{-1}^{1} x^2 f(x)\,dx$

4. Given $F(x) = (1 - \cos(x))/2 \quad x \in [0, \pi]$ find

a. ($\underline{1\ pt}$) $P(x < \frac{\pi}{2})$

($\underline{1pt}$) +1 for evaluating all 3 (will be diminished by calculation errors)

ANSWER: $P(x < \frac{\pi}{2}) = F(\frac{\pi}{2})$

1 pt for $p(x < \frac{\pi}{2}) = F(\frac{\pi}{2})$

$$= \frac{1 - \cos(\frac{\pi}{2})}{2}$$

$$= \frac{1 - 0}{2}$$

$$= \frac{1}{2}$$

b. ($\underline{1\ pt}$) $P(\frac{\pi}{4} < x < \frac{3\pi}{4})$

$$P(\frac{\pi}{4} < x < \frac{3\pi}{4}) = P(x < \frac{3\pi}{4}) - P(x < \frac{\pi}{4})$$

<-- 1pt for correctly partitioning $P(\frac{\pi}{4} < x < \frac{3\pi}{4})$

$$= F(\frac{3\pi}{4}) - F(\frac{\pi}{4})$$

$$= \frac{1 - \cos(\frac{3\pi}{4})}{2} - \frac{1 - \cos(\frac{\pi}{4})}{2}$$

$$= \frac{1 - (-\frac{1}{\sqrt{2}})}{2} - \frac{1 - \frac{1}{\sqrt{2}}}{2}$$

$$= \frac{\frac{2}{\sqrt{2}}}{2}$$

$$= \frac{1}{\sqrt{2}}$$

c. ($\underline{1\ pt}$) the pdf.

$$f(x) = \frac{d}{dx} F(x)$$

1 pt for $f(x) = \frac{d}{dx} F(x)$

$$= \frac{d}{d(x)} \frac{1 - \cos(x)}{2} = \frac{\sin(x)}{2}$$