Model Answers | Homework 5 | Stat 310

Total Points: 26

1. For each of the following random variables, identify the distribution that most closely matches the situation. Justify your choice and describe any assumptions that you made.

a. (4pts) On average, a flight from Houston to Dallas leaves every 60 minutes. Let X_1 be the amount of time I have to wait for the next airplane to come

ANSWER: $X_1 \sim \text{exponential}(60)$

1pt for exponential
1pt for $\theta = 60$

Reason: X_1 is the waiting time for a poisson r.v. (plane arrivals) with an average waiting time of 60.

1pt for a reason

Assumptions: Plane arrival times are independent of each other, etc.

1pt for an assumption

b. (4pts) 12oz. beer bottles have a mean volume of 11.8oz and a variance of 0.7oz. Let X_2 be the amount of beer in the bottle from this sample.

ANSWER: $X_2 \sim \text{normal}(11.8, 0.7)$
or truncated normal

1pt for normal
1pt for $\mu = 11.8$
$\sigma^2 = 0.7$
$(\sigma = 0.837)$

Reason X_2 is symmetrically distributed around 11.8oz (we'd assume) and has a higher chance of being close to 11.8oz than far from it.

1pt a reason

Assumptions - we must assume that a bottle could hold both $-\infty$ and ∞ oz. Or we must truncate our distribution to reasonable values.

1pt an assumption

c. (4pts). The registrar gets lazy and decides to use a random number generator to determine the threshold GPA for the presidents honor roll. Let X_3 be the minimum GPA to get you on the honor roll.

ANSWER

$$X_3 \sim \text{Uniform}(0.0, 4.0)$$

1pt uniform

1pt $a = 0$, $b = 4$

reason: each possible gpa has an equal chance of being selected. GPAs fall between 0 and 4.

1pt a reason

Assumption: the random number generator uses the same range as GPA's, it doesn't weight the possibility of choosing values, etc.

1pt an assumption

2. For each of the following random variables, find the specified probability using the CDF.

a. (2pts) $X_1 \sim \text{exponential}(\theta = 10)$ $P(10 < X_1 < 100)$

ANSWER: $P(10 < X_1 < 100) = F(100) - F(10)$

1pt translating probability to integral of CDF

$$= 1 - e^{-\frac{100}{10}} - \left(1 - e^{-\frac{10}{10}}\right)$$

1pt Correctly supplying CDF with $\theta = 10$

$$= 1 - e^{-10} - 1 + e^{-1}$$

$$= e^{-1} - e^{-10} \approx 0.368$$

2b (2pts) $X_2 \sim \text{Gamma}(\alpha=1, \beta=2)$, $P(1 < X_2 < 5)$

ANSWER: The CDF of the gamma distribution does not have a closed form. But that does not mean it cannot be calculated. For example, with Wolfram Alpha:

$$P(1 < X_2 < 5) = F(5) - F(1)$$ 1pt

$$= \text{CDF}[\text{gamma distribution}[1, \tfrac{1}{2}], 5] - \text{CDF}[\text{gamma} \ldots$$ 1pt

$$= \frac{1}{e^2} - \frac{1}{e^{10}} \approx 0.135$$

OR

Alternatively, students could integrate the pdf. We will reward full credit for doing this here, in part b, only.

$$P(1 < X_2 < 5) = \int_1^5 \frac{2^1}{\Gamma(1)} x^{1-1} e^{-2x}\, dx$$ +1pt

$$= \int_1^5 2e^{-2x}\, dx$$ +1pt

$$= \frac{e^8 - 1}{e^{10}} \approx 0.135$$

OR

Or Bonus students may recognize that $\text{Gamma}(\alpha=1, \beta=2)$ is the same as $\text{exponential}(\theta = \frac{1}{2})$ and use the exponential CDF

Bonus! +1pt for recognizing

$$P(1 < X_2 < 5) = F(5) - F(1)$$ +1pt

$$= 1 - e^{-2 \cdot 5} - (1 - e^{-2 \cdot 1})$$ +1pt

$$= e^{-2} - e^{-10} \approx 0.135$$

c. (2pts) $X_3 \sim \text{normal}(\mu = 0, \sigma^2 = 10)$, $P(-10 < X_3 < 10)$

ANSWER: The cdf of a normal distribution is not easily computed. However, we can look up values of a standard normal distribution in a probability table. To do that here, we should first convert X_3 to a standard normal r.v. ($\mu = 0, \sigma^2 = 1$)

$$P(-10 < X_3 < 10) = P\left(\frac{-10-\mu}{\sigma} < \frac{X_3 - \mu}{\sigma} < \frac{10-\mu}{\sigma}\right) = P\left(\frac{-10}{\sqrt{10}}\ Z < \frac{10}{\sqrt{10}}\right)$$ 1pt for converting to Z

$$= P(-\sqrt{10} < Z < \sqrt{10})$$

$$= \Phi(\sqrt{10}) - \Phi(-\sqrt{10})$$ 1pt for correctly interpretting probability

$$\approx 0.999 - 0.000782$$

$$\approx 0.998$$

3. Given that $X \sim Exponential(\theta)$, find the pdfs of the following two transformations of X. Do they correspond to named distributions that we know about?

a. (4pts) $Y = X^2$

notice that the range of the exponential distribution is $X \geq 0$.

Answer: $\therefore u(x) = x^2$ is invertible with inverse, $v(x) = \sqrt{x}$ — (1pt) for checking invertibility

Hence, we can use the change of variables formula

$$f_Y(y) = f_X(v(y))\,|v'(y)|$$

(1pt) for correctly defining $f_Y(y)$ in terms of $f_X(y)$

$$= \frac{1}{\theta} e^{-\frac{v(y)}{\theta}} |v'(y)|$$

$$= \frac{1}{\theta} e^{-\frac{\sqrt{y}}{\theta}} |v'(y)|$$

($\frac{1}{2}$pt) for correctly identifying $v(y)$ and $v'(y)$

$$= \frac{1}{\theta} e^{-\frac{\sqrt{y}}{\theta}} \frac{1}{2\sqrt{y}}, \; y \in [0, \infty)$$

(1pt) for correctly specifying range

Does not correspond to a known distribution — ($\frac{1}{2}$pt) for recognizing not a dist

ALTERNATIVE ANSWER: Or, we can use the distribution function technique

1. Figure out the range

$x \in [0, \infty) \Rightarrow Y = x^2 \in [0, \infty)$

2. Write the CDF for Y, then substitute for x

$$F_Y(y) = P(y < Y)$$

$$= P(x^2 < Y)$$

(1pt) for correctly specifying $F_Y(y)$ in terms of x

3. Find the region in X such that $u(x) < y$

$$P(x^2 < y) = P(-\sqrt{y} < x < \sqrt{y})$$

or

$$P(x < \sqrt{y}) \text{ because } y \in [0, \infty)$$

4. Integrate to compute CDF

$$P(-\sqrt{y} < x < \sqrt{y}) = \int_{-\sqrt{y}}^{\sqrt{y}} f(x)\,dx \quad \left(\text{or } \int_0^{\sqrt{y}} f(x)dx\right)$$

(1pt) for correctly transforming probability into an integral of x

$$= \int_{-\sqrt{y}}^{0} f(x)dx + \int_0^{\sqrt{y}} f(x)dx$$

$$= 0 + \int_0^{\sqrt{y}} f(x)dx \quad \text{because } f(x)=0\ \forall x<0$$

for an exponential dist.

$$= \int_0^{\sqrt{y}} \frac{1}{\theta} e^{-\frac{x}{\theta}} dy$$

$$= 1 - e^{-\frac{\sqrt{y}}{\theta}} \qquad y \in [0, \infty)$$

5. Differentiate to get the pdf

($\frac{1}{2}$pt) for finding the pdf by differentiating

$$\frac{d}{dy}\left(1 - e^{-\frac{\sqrt{y}}{\theta}}\right) = \frac{1}{2\theta\sqrt{y}} e^{-\frac{\sqrt{y}}{\theta}} \qquad y \in [0, \infty)$$

(1pt) for correctly specifying the range of y

This does not correspond to a known distribution

(1/2) for recognizing this

b. (4pts) $Z = e^X$

ANSWER 1: With the distribution function method (the steps are as listed above).

1.) $x \in [0, \infty) \quad y = e^x \in [1, \infty)$

2.) $F_Y(Y) = P(y < Y) = P(e^x < Y)$

(1pt) for correctly specifying $F_Y(Y)$ in terms of x

3.) $= P(x < \ln(y))$

4.) $= \int_0^{\ln(y)} f(x)dx$

(1pt) for correctly transforming probability into an integral

$$= \int_0^{\ln(y)} \frac{1}{\theta} e^{-\frac{x}{\theta}} dy$$

$$= 1 - e^{-\frac{\ln(y)}{\theta}}$$

5. $\frac{d}{dy}\left[1 - e^{-\frac{\ln y}{\theta}}\right] = \frac{1}{\theta} y^{-\left(\frac{\theta+1}{\theta}\right)}$ ($\frac{1}{2}$pt) for finding the pdf by differentiation

$\overset{or}{=} \frac{1}{\theta} y^{-\frac{1}{\theta}-1} \quad y \in [1, \infty)$ (1pt) for specifying the new range

This is not a distribution that we know ($\frac{1}{2}$pt) for recognizing not a known dist

Alternatively, students could use the change of variables method as $u(x) = e^x$ is invertible with $v(x) = \ln(x)$ (1pt) for checking invertibility

$$f_Y(y) = f_X(v(y))\,|v'(y)|$$ (1pt) for correctly defining $f_Y(y)$

$$= \frac{1}{\theta} e^{-\frac{v(y)}{\theta}} |v'(y)|$$ ($\frac{1}{2}$pt) for correctly identifying $v(y)$ and v'

$$= \frac{1}{\theta} e^{-\frac{\ln(y)}{\theta}} |v'(y)|$$

$$= \frac{1}{\theta} e^{-\frac{\ln(y)}{\theta}} \left|\frac{1}{y}\right|$$

$$= \frac{1}{\theta} y^{-\frac{1}{\theta}} y^{-1}$$

$$= \frac{1}{\theta} y^{-\frac{1}{\theta}-1} \qquad y \in [1, \infty)$$

($\frac{1}{2}$pt) for recognizing not a known dist

(1pt) for correctly specifying range

4 (Bonus 3pts) Show that if $X \sim Normal(\mu, \sigma^2)$ then $Z = (X-\mu)/\sigma$ has a standard normal distribution.

ANSWER: Recall from lecture, if $X \sim Normal(\mu, \sigma^2)$ and $Y = aX + b$, THEN $Y \sim Normal(a\mu + b, a^2\sigma^2)$ (**)

So, $Z = \frac{(X-\mu)}{\sigma} = \frac{1}{\sigma} X - \frac{\mu}{\sigma}$ (2pts) for solution shown

$\sim Normal\left(\frac{1}{\sigma}\mu - \frac{\mu}{\sigma}, \frac{1}{\sigma^2}\sigma^2\right)$ by (**) (3pts) for solving/proving with change of variables formula

$\sim Normal(0, 1)$