Homework 06 — Model Answers — Stat 310

Total Points: 19

1. Let $f(x,y) = c(x + 2xy + 2y)$, $x \in [0,1]$, $y \in y[0,1]$

a. (3 pts) What is c?

Answer: To be a pdf, $f(x,y)$ must integrate to 1 over its support and be strictly non-negative

$$\therefore \int_0^1 \int_0^1 f(x,y)\,dx\,dy = 1$$

(1pt) for recognizing this relationship

$$\int_0^1 \int_0^1 c(x + 2xy + 2y)\,dy\,dx = 1$$

$$\int_0^1 \left(cxy + cxy^2 + cy^2\right)\Big|_0^1 dx = 1$$

$$\int_0^1 2cx + c\,dx = 1$$

(1pt) for solving

$$cx^2 + cx\Big|_0^1 = 1$$

$$2c = 1$$

$$c = \frac{1}{2} = 0.5$$

(1pt) for checking that $f(x,y) \geq 0 \;\forall\; x,y \,|\, c = \frac{1}{2}$

$$f(x,y) = \frac{1}{2}(x + 2xy + 2y) \geq 0 \quad \forall\; x,y$$

b. (2 pts) What is $F(x,y)$?

Answer: The CDF $F(x,y) = P(X \leq x \text{ and } Y \leq y)$

$$= \int_{-\infty}^{x} \int_{-\infty}^{y} f(x,y)\,dy\,dx$$

$$= \int_0^x \int_0^y f(x,y)\,dy\,dx \quad \left(\text{because of range of } x \text{ and } y\right)$$

(1pt) for defining $F(x)$ as a bivariate integral with one of these two ranges

$$= \int_0^x \int_0^y \frac{1}{2}x + xy + y\;dy\,dx$$

$$= \int_0^x \frac{1}{2}xy + \frac{1}{2}xy^2 + \frac{1}{2}y^2\Big|_0^y dx$$

$$= \int_0^x \frac{1}{2}xy + \frac{1}{2}xy^2 + \frac{1}{2}y^2\,dx$$

$$= \frac{1}{4}x^2y + \frac{1}{4}x^2y^2 + \frac{1}{2}y^2x\Big|_0^x$$

(1pt) for solving correctly

$$= \frac{1}{4}x^2y + \frac{1}{4}x^2y^2 + \frac{1}{2}y^2x$$

2. Which of the following bivariate pdf's represent the pdf of two independent pdfs? (You can assume $x \in [0,1]$, $y \in [0,1]$ and you don't need to find the values of any constants.) Show your reasoning.

a. (2pts) $f(x,y) = C_1 e^{x+y}$

ANSWER: In general, X and Y are independent iff $f(x,y) = f_X(x) f_Y(y)$ so for a-d we can show independence by factoring the pdf into functions of x and y. If the pdf does not factor, we can conclude that x and y are not ind.

So. $$f(x,y) = C_1 e^{x+y}$$
$$= \underbrace{C_1 e^x}_{f_X(x)} \underbrace{e^y}_{f_Y(y)}$$

1pt for showing the pdf factors

1pt for clarifying that this proves independence

$\therefore$ X and Y are independent because $f(x,y) = f(x)f(y)$

b. (2pts) $f(x,y) = C_2 (x+y)$

ANSWER $f(x,y)$ cannot be factored into $f_X(x) f_Y(y)$

$\therefore$ X and Y are dependent

1pt for explaining that $f(x,y)$ cannot be factored

1pt for concluding dependence

c. (2pts) $f(x,y) = C_3 (xy + x + y + 1)$

ANSWER:
$$= \underbrace{C_3 (x+1)}_{f_X(x)} \underbrace{(y+1)}_{f(y)}$$

1pt for showing pdf factors

1pt for clarifying that this proves independence

$\therefore$ x and y are independent

d. (2 pts) $f(x,y) = C_4(x^2y^2 + x^2y + x^2)$

$$= \underbrace{C_4 x^2}_{f_X(x)} \underbrace{(y^2 + y + 1)}_{f_Y(y)}$$

1pt showing pdf factors

$\therefore$ X and Y are independent

1pt clarifying independence

3. $X \sim Unif(0,10)$, $Y|X = x \sim Exp(\theta = x)$. Find:

a. (2 pts) $f(x,y)$

ANSWER: $f(x,y) = f(x)f(y|x)$

$$= \frac{1}{10} \cdot \frac{1}{x} e^{-\frac{y}{x}}$$

$$= \frac{1}{10x} e^{-\frac{y}{x}}$$

1pt $f(x,y) = f(x)f(y|x)$

1pt for correctly identifing and substituting the pdfs of X and Y|X

b. (2 pts) $f(y)$

ANSWER: $f(y) = \int_x f(x,y)\,dx$

$$= \int_0^\infty \frac{1}{10} e^{-\frac{y}{x}}\,dx$$

= Something complex

1pt for $f(y) = \int_x f(x,y)dx$ or $\int_0^{10} f(x,y)dx$

1pt for substituting in $f(x,y)$ from above with correct limits

c. (2 pts) $E(Y)$

ANSWER: $E(Y) = \int_y y f(y)\,dy$

$$= \int_0^\infty y \int_0^{10} \frac{1}{10} e^{-\frac{y}{x}}\,dx\,dy$$

$$= \int_0^{10} \int_0^\infty \frac{y}{10} e^{-y/x}\,dy\,dx$$

$$= \frac{100}{3} = 33.\overline{3}$$

1pt for $E(Y) = \int_y y f(y)\,dy$

1pt for solving with correct limits on integration