Homework 07 Model Answers Garrett Grolemund

Total Points: 29

1. Let $X \sim Exp(1)$ and $Y \sim Exp(1)$. X and Y are independent.

a. (6 pts) Find the pdf of $A = \frac{X+Y}{2}$ and $B = \frac{X-Y}{2}$

ANSWER: By the bivariate transformation change of variables method

$$A = u_1(X,Y) = \frac{X+Y}{2} \qquad X = \frac{X+Y}{2} + \frac{X-Y}{2} = A+B = v_1(A,B)$$

$$B = u_2(X,Y) = \frac{X-Y}{2} \qquad Y = \frac{X+Y}{2} - \frac{X-Y}{2} = A-B = v_2(A,B)$$

1pt for $X=A+B, Y=A-B$

find the bounds: $x \geq 0$, $y \geq 0$; $a+b \geq 0$, $a-b \geq 0$; $b \geq -a$, $a \geq b$; $\therefore -a \leq b \leq a$

But also $x \geq 0$, $y \geq 0$; $a+b \geq 0$, $a-b \geq 0$; $a \geq -b$, $a \geq b$; $a \geq |b|$; $\therefore a \geq 0$

$a \in [0, \infty)$, $b \in [-a, a]$ or $a \in [|b|, \infty)$, $b \in (-\infty, \infty)$

$$J = \begin{vmatrix} \frac{\delta}{\delta a} X & \frac{\delta}{\delta b} X \\ \frac{\delta}{\delta a} Y & \frac{\delta}{\delta b} Y \end{vmatrix} = \begin{vmatrix} 1 & 1 \\ 1 & -1 \end{vmatrix} = -1 - 1 = -2$$

1pt Correctly setting up Jacobian (including correct method for solving determinant)

$$f_{X,Y}(x,y) = f_X(x) f_Y(y) \quad \text{(because X and Y are ind.)}$$

1pt for $f_{xy} = f_x \cdot f_y$

$$= 1e^{-x} \cdot 1e^{-y}$$

1pt for $f_x = e^{-x}$ and $f_y = e^{-y}$

$$= e^{-x} e^{-y}$$

$$= e^{-x-y} \qquad x,y \in [0, \infty)$$

$$f_{A,B}(a,b) = f_{X,Y}(v_1(a,b), v_2(a,b)) |J| \qquad **$$

$$= e^{-v_1(a,b) - v_2(a,b)} |-2|$$

$$= 2e^{-(a+b)-(a-b)}$$

1pt for correctly substituting in all terms in (**)

$$= 2e^{-a-b-a+b}$$

$$= 2e^{-2a} \qquad a \geq |b|, -\infty < b < \infty$$

$$a \geq 0, -a \leq b \leq a$$

1pt for correct bounds (one of the two shown)

b. (2pts) Are A and B independent?

ANSWER: A and B are not independent because the support of B depends on A.

1 pt - for not independe[nt]

1pt for the correct reason: support of B depends on A (or sample space is not a rectangle)

2. $f(x,y) = \theta e^{-(x+\theta y)}, x>0, y>0, (\theta>0)$

a. (7pts) find the pdf of $A = X \cdot Y$

$$A = u_1(X,Y) = X \cdot Y \qquad X = \frac{X \cdot Y}{Y} = \frac{a}{b} \quad \Big| B = Y = v_1(a,b)$$

$$B = u_2(X,Y) = Y \qquad Y = b = v_2(a,b)$$

1pt $X = \frac{a}{b}, Y = b$ (or $X = b, Y = \frac{a}{b}$)

note: if they do not choose $B = X$ or Y, make a note that it is simpler to set $B = X$ or $B = Y$

$$X > 0 \Rightarrow \frac{A}{B} > 0 \quad \Rightarrow A > 0$$

$$Y > 0 \quad \Rightarrow B > 0$$

$$J = \begin{vmatrix} \frac{\delta}{\delta a}\frac{a}{b} & \frac{\delta}{\delta b}\frac{a}{b} \\ \frac{\delta}{\delta a} b & \frac{\delta}{\delta b} b \end{vmatrix} = \begin{vmatrix} \frac{1}{b} & -\frac{a}{b^2} \\ 0 & 1 \end{vmatrix} = \frac{1}{b} - 0 = \frac{1}{b}$$

1pt correctly setting up Jacobian (includes using correct method to solve for determinant)

$$f_{A,B} = f_{X,Y}(v_1(a,b), v_2(a,b))\,|J|$$

$$= \theta e^{-\left(\frac{a}{b}+\theta b\right)} \cdot \frac{1}{b}$$

$$= \frac{\theta}{b} e^{-\left(\frac{a}{b}+\theta b\right)}$$

1pt for correctly substituting in all terms of C.O.V. formula.

1pt for $f_{A,B} = f_{X,Y}(v_1(a,b), v_2(a,b))|J|$

1pt for correct bounds on integration

$$f_A(a) = \int_0^\infty \frac{\theta}{b} e^{-\left(\frac{a}{b}+\theta b\right)} db \qquad a>0, b>0$$

1pt for setting up f_A as integral of $f_{A,B}$ with respect to b

1pt for $a>0, b>0$

= Something complex

b. (1pt) Is this a named distribution?

ANSWER: We don't know. But it is unlikely to be a named distribution, because named distributions usually result in integrable joint pdfs.

1pt for saying no or expressing doubt

3. Let $R \sim Unif(0,1)$ and $A \sim Unif(0, 2\pi)$. R and A are indep.

a. (8pts). Find the pdf of $X = R\cos(A)$ and $Y = R\sin(A)$

$X = u_1(R,A) = R\cos(A)$ $\quad R = v_1(x,y) = \sqrt{R^2}$

$Y = u_2(R,A) = R\sin(A)$

$= \sqrt{R^2(\cos^2 A + \sin^2 A)}$ $\quad$ (1pt) $R = \sqrt{x^2+y^2}$

$= \sqrt{(R\cos A)^2 + (R\sin A)^2}$

$= \sqrt{X^2+Y^2}$

$A = v_2(x,y) = \arctan\left(\frac{y}{x}\right)$ $\quad$ (1pt) $A = \arctan\left(\frac{y}{x}\right)$

because $\tan A = \frac{\sin A}{\cos A} = \frac{R\sin A}{R\cos A} = \frac{y}{x}$

bounds:

$0 \le R \le 1$ $\quad$ $0 \le A \le 2\pi$

$0 \le \sqrt{x^2+y^2} \le 1$ $\quad$ $0 \le \arctan(\frac{y}{x}) \le 2\pi$ $\quad \Rightarrow x, y$ such that $x^2+y^2 \le 1$

$0 \le x^2+y^2 \le 1$ $\quad$ $-\infty \le \frac{y}{x} \le \infty$

$x^2+y^2 \le 1$ $\quad$ not helpful

(1pt) taking correct partial derivatives

(1pt) correctly calculating determinant

$$J = \begin{vmatrix} \frac{dr}{dx} & \frac{dr}{dy} \\ \frac{dA}{dx} & \frac{dA}{dy} \end{vmatrix} = \begin{vmatrix} \frac{x}{\sqrt{x^2+y^2}} & \frac{y}{\sqrt{x^2+y^2}} \\ \frac{y}{x^2+y^2} & \frac{x}{x^2+y^2} \end{vmatrix} = \frac{x^2}{(x^2+y^2)^{3/2}} - \frac{y^2}{(x^2+y^2)^{3/2}} = \frac{x^2-y^2}{(x^2+y^2)^{3/2}}$$

$f_{R,A}(r,a) = 1 \cdot \frac{1}{2\pi}$ (because R and A are independent)

(1pt) for $f_r = \frac{1}{1}$, $f_a = \frac{1}{2\pi}$

(1pt) for $f_{r,a} = f_r \cdot f_a$ due to independence

$f_{X,Y}(x,y) = f_{R,A}(v_1(x,y), v_2(x,y))\,|J| = 1 \cdot \frac{1}{2\pi}|J|$

(1pt) for correctly substituting everything into change of variables formula

$$= \frac{1}{2\pi}\left|\frac{x^2-y^2}{(x^2+y^2)^{3/2}}\right|$$

$x^2+y^2 \le 1$

or

$-\sqrt{1-x^2} \le y \le \sqrt{1-x^2}$

$-1 \le x \le 1$

or

$-\sqrt{1-y^2} \le x \le \sqrt{1-y^2}$

$-1 \le y \le 1$

(1pt) correct bounds (any of 3 given)

$$\frac{\ln(0.10)}{\ln(0.9803)} \le n$$

$$115.727 \le n$$

$$n = 116 \text{ times}$$

1pt for 116 by this method. ($\frac{1}{2}$ point for 115.727)

5. ($\underline{3pts}$ - Bonus) Let $X_1, X_2, X_3, \dots X_n$ be independent with the same CDF, F. Find the pdf of $Y = \min(X_1, X_2, X_3, \dots X_n)$

ANSWER:

$$P(Y \le y) = 1 - P(Y > y) = 1 - P(X_1, \dots X_n > Y)$$

1pt for using $P(\min > a) = P(\text{all} > a)$

$$= 1 - P(X_1 > y)P(X_2 > y)\dots P(X_n > y)$$

because of independence

$\frac{1}{2}$pt for recognizing independence and separating the probability

$$= 1 - \left[(1 - P(X_1 \le y))(1 - P(X_2 \le y))\dots(1 - P(X_n \le y))\right]$$

$$= 1 - \left[(1 - F_{X_1}(y))(1 - F_{X_2}(y))\dots(1 - F_{X_n}(y))\right]$$

$\frac{1}{2}$ pt for recognizing that all $F_{X_i} = F_{X_1}$

$$= 1 - (1 - F_{X_1}(y))^n \quad \text{because } F_{X_1} = F_{X_2} = \dots F_{X_n}$$

which is the CDF of Y. To find the pdf take the derivative

$$f(y) = \frac{d}{dy}F(y) = \frac{d}{dy}\left(1 - (1 - F_{X_1}(y))^n\right)$$

1pt $f(y) = \frac{d}{dy}F(y)$ and solving

$$= n\frac{d}{dy}(F_{X_1}(y))(1 - F_{X_1}(y))^{n-1}$$

$$= n\,\underline{\underline{f_{X_1}(y)}}(1 - F_{X_1}(y))^{n-1} \quad \text{because } \frac{d}{dy}F_{X_1}(y) = f_{X_1}(y)$$

take off $\frac{1}{2}$ point for not simplifying $\frac{d}{dy}F_{X_1}(y) = f_{X_1}(y)$

4.c. (Alternative method: 1pt) How many times do you have to retake the quiz to have 90% chance of passing?

ANSWER: We notice the following pattern from parts a and b

$$P(\text{passing in } n \text{ tries}) = 1 - (1 - P(X_i \geq 6))^n$$

So...

$$.90 = 1 - (1 - P(X_i \geq 6))^n$$

$$(1 - P(X_i \geq 6))^n = 1 - .90$$

$$(1 - 0.0197)^n = .10$$

$$0.9803^n = 0.10$$

$$\ln(0.9803^n) = \ln(0.10)$$

$$n \ln(0.9803) = \ln(0.10)$$

$$n = \frac{\ln(0.10)}{\ln(0.9803)}$$

$$= 115.727$$

1pt for 116 by this method

($\frac{1}{2}$ point if they leave $n = 115.727$)

Since we can only take an integer # of tests, we must take 116 tests to guarantee a 90% chance of passing due to random chance.

4.b (Alternative method: 3pts) If you are allowed to take it twice and only take the highest score, what is the probability that you pass? what if you are allowed 3 times?

ANSWER: The highest score will be the max of the scores. So let

X_1 be the score on the 1^{st} test,
X_2 be the score on the 2^{nd} test.
Y be the $\max(X_1, X_2)$

then

(1pt) setting up $P(\max(X_1,X_2) \geq 6)$

$$P(\text{pass in 2 tries}) = P(Y \geq 6)$$
$$= 1 - P(Y < 6)$$
$$= 1 - P(\max(X_1,X_2) < 6)$$

(1pt) recognizing that max implies all <

$$= 1 - P(X_1 < 6 \cap X_2 < 6) \quad \text{(otherwise the max wouldn't be } \leq 6\text{)}$$
$$= 1 - P(X_1 < 6)P(X_2 < 6) \quad \text{(assuming independence)}$$
$$= 1 - (1 - P(X_1 \geq 6))(1 - P(X_1 \geq 6))$$
$$= 1 - (1 - P(X_1 \geq 6))^2 \quad (X_1 \text{ and } X_2 \text{ have the same probabilities})$$
$$= 1 - (1 - 0.0197)^2 \quad \text{(using results of part a)}$$
$$= 0.039$$

Let $Z = \max(X_1, X_2, X_3)$

$$P(\text{pass in 3 tries}) = P(Z \geq 6)$$
$$\vdots$$
$$= 1 - (1 - 0.0197)^3$$
$$\approx 0.058$$

(1pt) correctly inducing or reproving method for 3