Homework 08 Model Answers Stat 310

Total Points: 16

1. (4pts) Find Var(aX + bY) and express it as simply as possible (don't make any assumptions about X and Y).

ANSWER:

$$Var(aX+bY) = Var(aX) + Var(bY) + 2Cov(aX, bY)$$ (1pt) $Var(X+Y) = Var(X) + Var(Y) + 2Cov(X,Y)$

$$= a^2 Var(X) + b^2 Var(Y) + 2Cov(aX, bY)$$ (1pt) $Var(aX) = a^2 Var$

$$= a^2 Var(X) + b^2 Var(Y) + 2ab\,Cov(X, Y)$$ (1pt) $Cov(aX,Y) = a\,Cov(X,Y)$

(1pt) for right solution

2. (7pts) Let $X_1, X_2, \ldots X_n$ be iid Poisson(λ).

a. (3pts) What is the exact mgf of $S_n = \sum_{i=1}^{n} X_i$. Does this represent a named distribution? What is the mgf of $\bar{X}_n = \frac{S_n}{n}$?

ANSWER:

$$M_{S_n}(t) = E[e^{tS_n}] \quad \text{(def. of mgf)}$$

$$= E[e^{t(\sum_{i=1}^{n} X_i)}] \quad \text{(def of } S_n)$$

$$= E[e^{(tX_1 + tX_2 + \ldots tX_n)}]$$

$$= E[e^{tX_1} \cdot e^{tX_2} \cdots e^{tX_n}] \quad \text{(property of exponents)}$$

$$= E[e^{tX_1}]E[t^{X_2}] \cdots E[t^{X_n}] \quad (E[XY] = E[X]E[Y] \text{ if } X \text{ and } Y \text{ are independent})$$

$$= M_{X_1}(t) M_{X_2}(t) \cdots M_{X_3}(t) \quad \text{(def of mgf)}$$

$$= (M_{X_1}(t))^n \quad (X_i \text{ are identically distributed same distribution = same mgf})$$

$$= \left(e^{\lambda(e^t - 1)}\right)^n \quad \text{(mgf of poisson}(\lambda)\text{ r.v.)}$$

$$= e^{n\lambda(e^t - 1)} \; \square \quad \text{(property of exponents)}$$

$\Bigg\}$ (1pt) for correctly calculating mgf

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$$\text{mgf of } \bar{X}_n = E\left[e^{t\bar{X}_n}\right]$$
$$= E\left[e^{t\frac{S_n}{n}}\right]$$
$$= E\left[e^{\frac{t}{n}S_n}\right]$$
$$= E\left[e^{\frac{t}{n}X_1 + \frac{t}{n}X_2 + \dots \frac{t}{n}X_n}\right]$$
$$= E\left[e^{\frac{t}{n}X_1}\right]E\left[e^{\frac{t}{n}X_2}\right]\dots E\left[e^{\frac{t}{n}X_n}\right]$$
$$= M_{X_1}\left(\tfrac{t}{n}\right)M_{X_2}\left(\tfrac{t}{n}\right)\dots M_{X_n}\left(\tfrac{t}{n}\right)$$
$$= \left(M_{X_1}\left(\tfrac{t}{n}\right)\right)^n$$
$$= \left(e^{\lambda(e^{\frac{t}{n}}-1)}\right)^n$$
$$= e^{n\lambda(e^{\frac{t}{n}}-1)}$$

1pt

Cont'd: Is this a known distribution?

Let $\lambda' = n\lambda$, then

$Mgf\, S_n(t) = e^{\lambda'(e^t-1)}$ which is the mgf of a poisson(λ') r.v

$\therefore S_n \sim poisson(n\lambda)$

1pt for recognizing

b. (2pts) What is another mgf that $\bar{X}_n$ should be close to? Why?

The mgf of $\bar{X}_n$ should be close to the mgf for a normal$\left(\lambda, \frac{\lambda}{n}\right)$ random variable

$\frac{1}{2}$pt normal

$\frac{1}{2}$pt $\mu = \lambda$ $\sigma^2 = \frac{\lambda}{n}$

... this is because for large n the mean of an iid random sample $\Rightarrow$ normal$\left(\mu, \frac{\sigma^2}{n}\right)$ by the central limit theorem

1pt reason is the CLT

c. (2pts) Compare the mgf of exact distribution (from a) to the approximate distribution (from b). What happens as $n \to \infty$?

ANSWER: $\lim_{n\to\infty} M_{\bar{X}_n}(t) = \lim_{n\to\infty} e^{n\lambda(e^{\frac{t}{n}}-1)}$

$= e^{t\lambda}$ W.A.

$\lim_{n\to\infty} M_{N(\lambda,\frac{\lambda}{n})}(t) = \lim_{n\to\infty} e^{(\lambda t + \frac{1}{2}\frac{\lambda}{n}t^2)}$

1pt mgf of a normal $(\lambda, \frac{\lambda}{n})$ r.v

$= e^{t\lambda}$ W.A.

1pt for showing that the limits both go to $e^{t\lambda}$ as $n \to \infty$

Althoug the mgf look different, they Converge to the same value in the limit

3. (5pts) Let X be a random variable with pdf

$f(x) = 630x^4(1-x)^4, \quad x \in (0,1)$

a. (2pts) Find $E(x)$ and $Var(x)$

ANSWER: $E(x) = \int_0^1 x f(x)\,dx$

$= \int_0^1 x\, 630x^4(1-x)^4\,dx$

$= \int_0^1 630x^5(1-x)^4\,dx$

$= \frac{1}{2}$ 1pt

$$\begin{aligned}
Var(x) &= E(x^2) - E(x)^2 \\
&= E(x^2) - \tfrac{1}{4} \\
&= \int_0^1 x^2\, 630x^4(1-x)^4\,dx - \tfrac{1}{4} \\
&= \int_0^1 630x^6(1-x)^4\,dx - \tfrac{1}{4} \\
&= \tfrac{3}{11} - \tfrac{1}{4} \\
&= \tfrac{1}{44} \approx 0.0227
\end{aligned}$$

1pt

b. (2pts) Obtain the lower bound given by Chebyshev's inequality for $P(0.2 < X < 0.8)$

ANSWER:

$$\begin{aligned}
P(0.2 < x < 0.8) &= P\left(\left|x - \tfrac{1}{2}\right| < 0.3\right) \\
&= P(|x - E(x)| < 0.3) \\
&= P\left(|x - E(x)| < \tfrac{0.3}{\sqrt{1/44}} \cdot \sqrt{1/44}\right) \\
&= P\left(|x - E(x)| < 1.99\sqrt{Var(x)}\right) \\
&= 1 - P\left(|x - E(x)| \geq 1.99\sqrt{Var(x)}\right) \\
&\geq 1 - \frac{1}{(1.99)^2} \quad \text{(by Chebyshev's } \neq \text{)} \\
&\geq 0.747 \text{ (approximately)}
\end{aligned}$$

1pt for $k = 1.99$ or $\frac{0.3}{\sqrt{Var(x)}}$

1pt for taking probability of the complement to use chebyshev's

c. (1pt) Compute the exact $P(0.2 < x < 0.8)$

ANSWER:

$$P(0.2 < x < 0.8) = \int_{0.2}^{0.8} f(x)\,dx = \int_{0.2}^{0.8} 630x^4(1-x)^4\,dx$$

$$= 0.961 \quad \text{W.A.}$$

1pt for using correct integral w/ correct bounds