Homework 9 Model Answers Stat 310

Total points: 14

1. (6pts) Using method of moments,

a. Find the estimators for Binomial(n,p)

$E(x) = np \qquad Var(x) = np(1-p)$

i. $\frac{Var(x)}{E(x)} = \frac{np(1-p)}{np}$

$p = 1 - \frac{Var(x)}{E(x)}$

$\boxed{p = 1 - \frac{S_n^2}{\bar{X}}}$

ii. $E(x) = np$

$n = \frac{E(x)}{p}$

$\boxed{n = \frac{\bar{X}}{1 - \frac{S_n^2}{\bar{X}}} = \frac{\bar{X}^2}{\bar{X} - S_n^2}}$

1pt for p

1pt for n

b. Find the estimators for Normal(μ, σ^2)

$E(x) = \mu \qquad Var(x) = \sigma^2$

i. $E(x) = \mu \qquad Var(x) = \sigma^2$

$\bar{X} = \mu \qquad S_n^2 = \sigma^2$

$\boxed{\mu = \bar{X}} \qquad \boxed{\sigma^2 = S_n^2}$

1pt for μ

1pt for σ^2

c. For Exponential(θ), find two methods of moments est. for θ.

$E(x) = \theta$

$\bar{X} = \theta$

$\boxed{\theta = \bar{X}}$

$Var(x) = \theta^2$

$\sqrt{Var(x)} = \theta$

$\boxed{\theta = \sqrt{S_n^2}}$

1pt for $\theta = \bar{x}$

1pt for $\theta = \sqrt{S_n^2}$

2. (4pts) An estimator of λ when $X_i \overset{iid}{\sim}$ Poisson(λ) is $\frac{\sum X_i}{n}$

a. Is this estimator unbiased?

$\hat{\lambda} = \frac{1}{n}\sum X_i$ will be unbiased iff $E(\hat{\lambda}) = \lambda$

$$
\begin{aligned}
E(\hat{\lambda}) &= E\left(\tfrac{1}{n}\sum X_i\right) \\
&= \tfrac{1}{n}E\left(\sum X_i\right) \\
&= \tfrac{1}{n}\sum E(X_i) \quad \text{because } X_i \text{ are independent} \\
&= \tfrac{1}{n}\sum \lambda \\
&= \tfrac{n}{n}\lambda \\
&= \lambda
\end{aligned}
$$

1pt for proving $E(\hat{\lambda}) = \lambda$

$\therefore\ \hat{\lambda} = \frac{1}{n}\sum X_i$ is unbiased

1pt for concluding unbiased.

b. Does it converge to λ? Why?

Yes.

1pt

By the Law of Large Numbers,

$\hat{\lambda} = \frac{1}{n}\sum X_i$ conveges in probability to $E(x) = \lambda$ 1pt

as $n \to \infty$

3. (4 pts) Find the MLE of θ when X_i come from an iid exponential distribution

$$f(x) = \frac{1}{\theta} e^{-x/\theta}$$

$$f(x_1, x_2, \dots x_n) = \prod_{i=1}^{n} \frac{1}{\theta} e^{-x_i/\theta}$$

$$= \frac{1}{\theta^n} e^{-\frac{1}{\theta}\sum_{i=1}^{n} x_i}$$

1pt setting up joint pdf

$$\ell(\theta | x_1, x_2, \dots x_n) = \ln\left(\frac{1}{\theta^n} e^{-\frac{1}{\theta}\sum_{i=1}^{n} x_i}\right)$$

$$= -\ln(\theta^n) + \ln\left(e^{-\frac{1}{\theta}\sum_{i=1}^{n} x_i}\right)$$

$$= -n\ln\theta - \frac{1}{\theta}\sum_{i=1}^{n} x_i$$

1pt seeking the maximum of the likelihood or log likelihood function

$$\max_{\theta} \ell(\theta | x_i) = \theta \text{ s.t. } \frac{d}{d\theta}\ell(\theta|x_i) = 0 \text{ ; } \frac{d^2}{d\theta^2}\ell(\theta|x_i) \leq 0$$

$$\frac{d}{d\theta}\ell(\theta|x_i) = \frac{\sum x_i}{\theta^2} - \frac{n}{\theta} = 0$$

$$\frac{\sum x_i}{\theta^2} = \frac{n}{\theta}$$

$$\boxed{\theta = \frac{\sum x_i}{n}}$$

1pt finding that $\frac{\sum x_i}{n}$ maximizes ℓ

$$\frac{d^2}{d\theta^2}\ell(\theta|x_i) = \frac{n}{\theta^2} - 2\frac{\sum x_i}{\theta^3}$$

$$\frac{d^2}{d\theta^2}\ell\left(\tfrac{1}{n}\textstyle\sum x_i \middle| x_i\right) = \frac{n}{\left(\frac{1}{n}\sum x_i\right)^2} - \frac{2\sum x_i}{\left(\frac{1}{n}\sum x_i\right)^3}$$

$$= \frac{n^3}{(\sum x_i)^2} - \frac{2n^3\sum x_i}{(\sum x_i)^3}$$

$$= \frac{-n^3}{(\sum x_i)^2} \leq 0 \quad \forall\, n > 0, \ \forall\, x_i$$

1pt checking that it is a maximum

$\therefore\ \hat{\theta} = \frac{\sum x_i}{n} = \bar{x}$ maximizes the likelihood and is the maximum likelihood estimator