Stat310

Bivariate transformations 2

Hadley Wickham

1. Change of variables technique
2. Bivariate normal distribution

Bivariate transformation

Example

Oscar has a bad gambling problem. Every night on the way home from work he takes the X hundred dollars he earned at work that day and goes to the local casino. Oscar never wins any money but eventually stops playing to return home with Y hundred dollars.

$$f(x,y) = \frac{1}{8} \quad 0 < y < x < 4$$

$$f(x) = \frac{x}{8} \quad 0 < x < 4$$

$$f(y) = \frac{1}{8}(4-y) \quad 0 < y < 4$$

Questions

A != (X - Y)/Y

Let A = (X - Y)/X and B = X. Are A and B independent? How can we tell?

Need to find pdf of A and B. Make two new random variables from two existing random variables: **transformation**

2d transformations

Distribution function technique

(hardly ever works, but easy)

Change of variable technique

(usually works, moderately hard)

2d transformations

Distribution function technique

(hardly ever works, but easy)

Change of variable technique

(usually works, moderately **easy**)

Steps

Write down u_1, u_2, $f_{X,Y}$ (simple)

Figure out v_1 and v_2 (90% simple)

Figure out bounds of A and B (simple)

Compute partial derivatives (simple)

Plug into formula (very simple)

```
# Figure out v1 and v2
# with wolfram alpha

a = (x - y) / x, b = x

a = (x - y) / x, b = x,
  x > 0, y > 0, x < 4, y < 4, y < x
```

Figure out bounds

- Convert complex bound into sequence of simpler bounds
- Transform each bound
- Simplify

Be careful of two different uses of | |

$$J = \left| \begin{matrix} \frac{\delta x}{\delta a} & \frac{\delta x}{\delta b} \\ \frac{\delta y}{\delta a} & \frac{\delta y}{\delta b} \end{matrix} \right|$$

$$\left| \begin{matrix} a & b \\ c & d \end{matrix} \right| = ad - bc$$

d/da b, d/db b , d/da b - ab, d/db b-ab

2d change of variables

$$A = u_1(X, Y) \quad X = v_1(A, B)$$

$$B = u_2(X, Y) \quad Y = v_2(A, B)$$

$$f_{A,B} = f_{X,Y}(v_1(A, B), v_2(A, B))|J|$$

Questions

- What is the distribution of U and V?
- Are U and V independent?
- What is the distribution of U?
  (But you'll never have to do a non-rectangular integral)

Practice

$f(x, y) = x + y \qquad 0 < x, y < 1$

$A = X + Y, B = X - Y$

Find v_1 and v_2, and the range of A, B

(Everything else is just plug and chug)

Your turn

$f(x, y) = x + y \qquad 0 < x, y < 1$

$A = XY, B = X$

Find v_1 and v_2, and the range of A, B

Bivariate normal

demonstrations.wolfram.com/
JointDensityOfBivariateGauss
ianRandomVariables/

$$f(x,y) = \frac{1}{2\pi\sigma_x\sigma_y\sqrt{1-\rho^2}} \exp\left[-\frac{q(x,y)}{2}\right]$$

$$q(x,y) = \frac{1}{1-\rho^2}\left[z_x^2 + z_y^2 - 2\rho z_x z_y\right]$$

$$z_x = \frac{x-\mu_x}{\sigma_x} \qquad z_y = \frac{x-\mu_y}{\sigma_y}$$

$$\frac{1}{2\pi^{-2k}\sqrt{|\Sigma|}}\exp(-\frac{1}{2}(x-\mu)'\Sigma^{-1}(x-\mu))$$

Independence

If $\rho = 0$, what does that imply about X and Y?

So

Independence implies correlation = 0.

If (and only if) bivariate normal, correlation = 0 implies independence.

Marginal and conditionals

Both marginal and conditional distributions are normal.

$$\text{X} \sim \text{Normal}(\mu_x, \sigma_x^2) \qquad \text{Y} \sim \text{Normal}(\mu_y, \sigma_y^2)$$

$$\text{X}|\text{Y} \sim \text{Normal}(\mu_x + \rho\frac{\sigma_x}{\sigma_y}(y - \mu_y), \sigma_x^2(1-\rho^2))$$

$$\text{Y}|\text{X} \sim \text{Normal}(\mu_y + \rho\frac{\sigma_y}{\sigma_x}(x - \mu_x), \sigma_y^2(1-\rho^2))$$

Change of variables

Let X and Y be independent standard normals.

Let A = X - Y and B = X + Y.

What is the joint distribution of A and B?

Solving the problem

- What are u_1 and u_2?
- What are ranges of A and B?
- What are v_1 and v_2?
- What is $|J|$?
- What is $f_{A,B}(a, b)$?

WA

a = x - y, b = x + y

domain and range (a + b)/2

local extrema (b - a)/2

da (a+b)/2, da (b-a)/2, db (a+b)/2, db (b - a) /2