PRELIMINARIES.

* $M_X(0) = E(e^{X0}) = E(1) = 1$

* Taylor series

$$f(x) = f(a) + (x-a)f'(a) + \frac{(x-a)^2}{2!}f''(a) + \frac{(x-a)^3}{3!}f''(a) + \ldots$$

more precisely can write ... as ε_x and (it's known that $\lim_{x \to a} \varepsilon_x = 0$ (i.e. as x gets closer to a the error decreases)

and even more strongly $\lim_{x \to a} \frac{\varepsilon_x}{(x-a)^2} = 0$

* Taylor series expansion of $M_X(t)$ at $t=0$ is:

$$M_X(t) = M_X(0) + (t-0)M_X'(0) + \frac{(t-0)^2}{2!}M_X''(0) + \varepsilon_t$$

$$= 1 + tE(X) + \frac{t^2}{2}E(X^2) + \varepsilon_t \qquad \left(\lim_{t \to 0} \frac{\varepsilon_t}{t^2} = 0\right)$$

* Random mathematical fact:

If $\lim_{n \to \infty} a_n = a$ then $\lim_{n \to \infty} \left(1 + \frac{a_n}{n}\right)^n = e^a$

CENTRAL LIMIT THEOREM

Let $X_1, X_2, \ldots$ be iid random variables with $E(X) = 0$ and $Var(X) = \sigma^2$

(Therefor $E(X^2) = \sigma^2$ because $Var(X) = E(X^2) - E(X)^2$)

Let $S_n = \sum_{i=1}^{n} X_i$

$$Z_n = \frac{S_n}{\sqrt{n}\,\sigma}$$ Then $\lim_{n\to\infty} Z_n = Z \sim \text{Normal}(0,1)$

Steps: find (approximate) $M_X(t)$, then $M_{S_n}(t)$, then $M_{Z_n}(t)$, then take limits

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$$M_X(t) = 1 + t\,E(X) + \frac{t^2}{2}E(X^2) + \varepsilon$$ (Talyor series)

$$= 1 + \frac{t^2\sigma^2}{2} + \varepsilon_s$$

$$M_{S_n}(t) = \left(M_X(t)\right)^n$$ because X's are iid

$$= \left(1 + \frac{t^2\sigma^2}{2} + \varepsilon_s\right)^n$$

$$M_{Z_n}(t) = M_{S_n}\left(\frac{t}{\sqrt{n}\,\sigma}\right)$$

$$= \left(1 + \left(\frac{t}{\sqrt{n}\,\sigma}\right)^2 \frac{\sigma^2}{2} + \varepsilon_{t/\sqrt{n}\sigma}\right)^n$$

$$= \left(1 + \frac{t^2}{2n} + \varepsilon_n\right)^n$$

there are some technical details here that we ignore, but if we did then we'd know $\lim_{n\to\infty} n\varepsilon_n = 0$.

$$M_{Z_n}(t) = \left(1 + \frac{t^2/2 + n\varepsilon_n}{n}\right)^n$$

$$\left[\ \lim_{n\to\infty} \frac{t^2}{2} + n\varepsilon_n = \frac{t^2}{2} + \lim_{n\to\infty} n\varepsilon_n = \frac{t^2}{2}\ \right]$$

$\Rightarrow \lim_{n\to\infty} Z_n(t) = e^{t^2/2}$ by random mathematical fact.

$\Rightarrow \lim_{n\to\infty} Z_n(t) = Z \sim \text{Normal}(0,1)$ by comparing mgfs

$\square$