MLE Variance

1. Joint pdf / likelihood $f(\underset{\sim}{x})$
2. Log-likelihood $\ell(\underset{\sim}{x})$
3. Differentiate & set $= 0$
4. (Check is maximum)

For normal

$$f(x_i) = \frac{1}{\sqrt{2\pi}\,\sigma} \exp\left(-\frac{(x_i-\mu)^2}{2\sigma^2}\right)$$

$$f(\underset{\sim}{x}) = \prod_{i=1}^{n} \frac{1}{\sqrt{2\pi}\,\sigma} \exp\left(-\frac{(x_i-\mu)^2}{2\sigma^2}\right)$$

$$= (2\pi)^{-n/2}\, \sigma^{-n} \exp\left(-\sum_{i=1}^{n}\frac{(x_i-\mu)^2}{2\sigma^2}\right)$$

$$\ell(\underset{\sim}{x}) = -\frac{n}{2}\ln 2\pi - n\ln\sigma - \sum_{i=1}^{n}\frac{(x_i-\mu)^2}{2\sigma^2}$$

$$\frac{d\ell(x)}{d\sigma} = \frac{-n}{\sigma} + \frac{\sum_{i=1}^{n}(x_i-\mu)^2}{\sigma^3} = 0$$

$$\frac{\sum(x_i-\mu)^2}{\sigma^2} = n$$

$$\hat{\sigma}^2 = \frac{\sum(x_i-\mu)^2}{n}$$

But don't know what μ is:

$$\Rightarrow \hat{\sigma}^2 = \frac{\sum(X_i-\bar{X}_n)^2}{n} = \boxed{S_n^2}$$

note it's ok to use upper or lower case ⇒ directs our thinking to single number of dist.

$$E(\hat{\sigma}^2) = \frac{1}{n}\sum_{i=1}^{n} E(X_i^2) - \frac{1}{n^2}\sum_{i=1}^{n}\sum_{j=1}^{n} E(X_i X_j)$$

$$\mathrm{Var}(X_i^2) = E(X_i^2) - E(X_i)^2 \Rightarrow E(X_i^2) = \sigma^2 + \mu^2$$

$$\mathrm{Cov}(X,Y) = E(XY) - E(X)E(Y) \Rightarrow E(XY) = \mu^2 \quad (X \neq Y)$$

$$= \frac{1}{n}\, n(\sigma^2+\mu^2) - \frac{1}{n^2}$$

$$\begin{bmatrix} X_1X_1 & X_2X_1 & \\ X_2X_1 & X_2X_2 & \\ & & \ddots & \\ & & & X_nX_n \end{bmatrix} \quad \begin{matrix} n(\sigma^2+\mu^2) + \\ (n^2-n)\mu^2 \end{matrix}$$

$$= \sigma^2 + \mu^2 - \frac{1}{n}\left(\sigma^2 + \cancel{\mu^2} + n\mu^2 - \cancel{\mu^2}\right)$$

$$= \sigma^2 - \frac{\sigma^2}{n} = \frac{n-1}{n}\sigma^2$$

Five fun facts about χ^2

$A \sim \chi^2(a) \qquad B \sim \chi^2(b)$

1) $E(A) = a$
2) $Var(A) = 2a$
3) $A + B \sim \chi^2(a+b)$
4) $A - B \sim \chi^2(a-b)$ if $a > b$
5) $Z \sim N(0,1) \Rightarrow Z^2 \sim \chi^2(1)$

Variance CI

1) $X = \dfrac{9\hat{s}^2}{\sigma^2} \sim \chi^2(9)$

2) $P(a < X < b) = 0.95$

$P(2.7 < X < 19) = 0.975 - 0.025 = 0.95$

3) $P\left(2.7 < \dfrac{9\hat{s}^2}{\sigma^2} < 19\right)$

4) $\Rightarrow P\left(1/2.7 > \dfrac{\sigma^2}{9\hat{s}^2} > 1/19\right)$

$\Rightarrow P\left(9/2.7\,\hat{s}^2 > \sigma^2 > 9/19\,\hat{s}^2\right)$

$P(3.3s^2 > \sigma^2 > 0.47)$

4) $\hat{\sigma}^2 \in (0.47s^2, 3.3s^2)$

$\in (0.26, 1.82)$