

## MLE Variance

1. Joint pdf/likelihood  $f(x_2)$
2. Log-likelihood  $l(x_2)$
3. Differentiate & set = 0
4. (check is maximum)

For normal

$$f(x_1) = \frac{1}{\sqrt{2\pi}\sigma} \exp\left(-\frac{(x_1 - \mu)^2}{2\sigma^2}\right)$$

$$f(x_2) = \prod_{i=1}^n \frac{1}{\sqrt{2\pi}\sigma} \exp\left(-\frac{(x_i - \mu)^2}{2\sigma^2}\right)$$

$$= (2\pi)^{-n/2} \sigma^{-n} \exp\left(-\sum_{i=1}^n \frac{(x_i - \mu)^2}{2\sigma^2}\right)$$

$$l(x) = -\frac{n}{2} \ln 2\pi - n \ln \sigma - \sum_{i=1}^n \frac{(x_i - \mu)^2}{2\sigma^2}$$

$$\frac{dl(x)}{d\sigma} = -\frac{n}{\sigma} + \frac{\sum_{i=1}^n (x_i - \mu)^2}{\sigma^3} = 0$$

$$\frac{\sum (x_i - \mu)^2}{\sigma^2} = n$$

$$\hat{\sigma}^2 = \frac{\sum (x_i - \mu)^2}{n}$$

But don't know what  $\mu$  is:

$$\Rightarrow \hat{\sigma}^2 = \frac{\sum (x_i - \bar{x}_n)^2}{n} = \boxed{S^2}$$

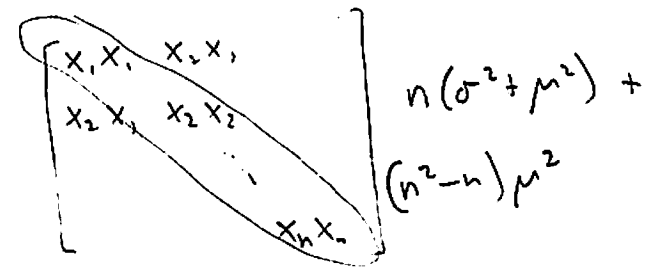
note it's ok to use upper or lower case  $\Rightarrow$  directs our thinking to single num. of dist.

$$E(\hat{\sigma}^2) = \frac{1}{n} \sum_{i=1}^n E(X_i^2) - \frac{1}{n^2} \sum_{i=1}^n \sum_{j=1}^n E(X_i X_j)$$

$$\text{Var}(X_i^2) = E(X_i^2) - E(X_i)^2 \Rightarrow E(X_i^2) = \sigma^2 + \mu^2$$

$$\text{Cov}(X, Y) = E(XY) - E(X)E(Y) \Rightarrow E(XY) = \mu^2 \quad (X \neq Y)$$

$$= \frac{1}{n} n(\sigma^2 + \mu^2) - \frac{1}{n^2}$$



$$= \sigma^2 + \mu^2 - \frac{1}{n} (\sigma^2 + \mu^2 + n\mu^2 - \mu^2)$$

$$= \sigma^2 - \frac{\sigma^2}{n} = \frac{n-1}{n} \sigma^2$$

## Five Fun Facts about $\chi^2$

$$A \sim \chi^2(a) \quad B \sim \chi^2(b)$$

$$1) E(A) = a$$

$$2) \text{Var}(A) = 2a$$

$$3) A + B \sim \chi^2(a+b)$$

$$4) A - B \sim \chi^2(a-b) \text{ if } a > b$$

$$5) Z \sim N(0,1) \Rightarrow Z^2 \sim \chi^2(1)$$

## Variance CI

$$1) X = \frac{9\hat{\sigma}^2}{\sigma^2} \sim \chi^2(9)$$

$$2) P(a < X < b) = 0.95$$

$$P(2.7 < X < 19) = 0.975 - 0.025 = 0.95$$

$$3) P(2.7 < \frac{9\hat{\sigma}^2}{\sigma^2} < 19)$$

$$4) \Rightarrow P(\frac{1}{2.7} > \frac{\sigma^2}{9\hat{\sigma}^2} > \frac{1}{19})$$

$$\Rightarrow P(\frac{9}{2.7}\hat{\sigma}^2 > \sigma^2 > \frac{9}{19}\hat{\sigma}^2)$$

$$P(3.35\hat{\sigma}^2 > \sigma^2 > 0.47\hat{\sigma}^2)$$

$$4) \hat{\sigma}^2 \in (0.47s^2, 3.35s^2)$$

$$\in (0.26, 1.82)$$