STAT310
Practice Problems
Week 2

February 9, 2012

1 Mutually independent events.

For this problem, note that three events $A_j,\ j = 1, 2, 3$ are **mutually independent** if (1) $P(A_i \cap A_j) = P(A_i)P(A_j)\ \forall i \neq j$ (pairwise independent), and (2) $P(A_1 \cap A_2 \cap A_3) = P(A_1)P(A_2)P(A_3)$.

(a)

$$\begin{aligned}
P(A \cap (B \cap C)) &= P(A \cap B \cap C) \\
&= P(A)P(B)P(C)\ (\textit{mutually independent}) \\
&= P(A)P(B \cap C)\ (\textit{mutual independence} \implies \textit{pairwise independence})
\end{aligned}$$

(b)

$$\begin{aligned}
P(A \cap (B \cup C)) &= P((A \cap B) \cup (A \cap C)) \\
&= P(A \cap B) + P(A \cap C) - P((A \cap B) \cap (A \cap C)) \\
&= P(A)P(B) + P(A)P(C) - P(A \cap B \cap C) \\
&= P(A)P(B) + P(A)P(C) - P(A)P(B)P(C) \\
&= P(A)[P(B) + P(C) - P(B)P(C)] \\
&= P(A)P(B \cup C)
\end{aligned}$$

(c)
Put $D = B \cap C$ and partition it using A:

$$D = (D \cap A) \cup (D \cap \bar{A}).$$

Then

$$\begin{aligned}
& P(D) = P(D \cap A) + P(D \cap \bar{A}) \\
\implies\ & P(D \cap \bar{A}) = P(D) - P(D \cap A).
\end{aligned}$$

Substituting $D = B \cap C$:

$$\begin{aligned} P(\bar{A} \cap B \cap C) &= P(B \cap C) - P(B \cap C \cap A) \\ &= P(B \cap C) - P(B \cap C)P(A) \text{ (\textit{mututal independence})} \\ &= P(B \cap C)[1 - P(A)] \\ &= P(B \cap C)P(\bar{A}) \end{aligned}$$

(d)

First note that $P(A \cap (\bar{B} \cap C)) = P((A \cap C) \cap \bar{B})$ due to commutativity of sets. Therefore, showing independence of A and $\bar{B} \cap C$ is equivalent to showing independence of $A \cap C$ and $\bar{B}$.

Put $D = A \cap C$ and partition it using B:

$$D = (D \cap \bar{B}) + (D \cap B).$$

Then

$$\begin{aligned} & P(D) = P(D \cap \bar{B}) + P(D \cap B) \\ \Longrightarrow\ & P(D) = P(D \cap \bar{B}) + P(D \cap B) \\ \Longrightarrow\ & P(D \cap \bar{B}) = P(D) - P(D \cap B) \end{aligned}$$

Substituting $D = A \cap C$:

$$\begin{aligned} P(A \cap C \cap \bar{B}) &= P(A \cap C) - P(A \cap C \cap B) \\ &= P(A \cap C) - P(A \cap C)P(B) \\ &= P(A \cap C)(1 - P(B)) \\ &= P(A \cap C)P(\bar{B}) \end{aligned}$$

2 Mammography.

(a)

	Breast Cancer	No Breast Cancer
Positive Mammogram	14	96
Negative Mammogram	0	890

For ease of notation, let us denote

D+ = Breast cancer
D- = No breast cancer
T+ = Positive mammogram
T- = Negative mammgram

Thus

$$\begin{aligned} P(D + |T+) &= \frac{14}{14 + 96} \\ &= 12.3\% \end{aligned}$$

(b) Bayes' rule tells us that

$$\begin{aligned} P(D+|T+) &= \frac{P(T+|D+)P(D+)}{P(T+|D+)P(D+)+P(T+|D-)P(D-)} \\ &= \frac{[1-P(T-|D+)]P(D+)}{[1-P(T-|D+)]P(D+)+P(T+|D-)[1-P(D+)]}. \end{aligned}$$

For women in their 50's:

$$\begin{aligned} P(D+|T+) &= \frac{(1-0.0011)(\frac{1}{38})}{(1-0.0011)(\frac{1}{38})+(0.0866)(1-\frac{1}{38})} \\ &= 23.8\%. \end{aligned}$$

For women in their 60's:

$$\begin{aligned} P(D+|T+) &= \frac{(1-0.0014)(\frac{1}{27})}{(1-0.0014)(\frac{1}{27})+(0.079)(1-\frac{1}{27})} \\ &= 32.7\%. \end{aligned}$$

Notice how low the probability is of a patient having breast cancer given that they have a positive mammography result! This low positive predictive value (see below) is one of the reasons that NIH recommendations for when to begin mammography screening women for breast cancer was recently pushed back. For more information on this topic, see

http://www.cancer.gov/cancertopics/pdq/screening/breast/healthprofessional/page6

Side note. This quantity that we were asked to calculate is called the **positive predictive value (PPV)**. In general, some commonly encountered quantities you may encounter in disease screening articles (such as the one above) include:

Sensitivity = $P(T+|D+)$
Specificity = $P(T-|D-)$
Positive Predictive Value (PPV) = $P(D+|T+)$
Negative Predictive Value (NPV) = $P(D-|T-)$

Along with other measures, these parameters are often used by clinicians and biostatisticians to guide recommendations on standard of care screening practices.

(c) Mammography involves exposure to radiation, which is a known carcinogen. A breast cancer biopsy involves approximately 6 times the amount of radiation used in one mammogram. Pre-menopausal women have hormonal protection against breast cancer and most women enter menopause in their 50s. Since the high false positive rate for women in their 40s, 9.78%, leads to more biopsies and significantly more exposure to radiation, the question is whether mammograms for these women lead to statistically significantly higher cancer rates because of overtreatment.

(d) This would require $P(D-|T+) < 0.10$. From Bayes' rule, we can see that $P(D-|T+) \propto P(T+|D-)P(D-)$. We can see from here that higher prevalence of the disease (equivalent to lower $P(D-)$) would decrease $P(D-|T+)$ (the probability of a false positive).

(e) This would require $P(D+|T-) < 0.10$. From Bayes' rule, we can see that $P(D+|T-) \propto P(T-|D+)P(D+)$. We can see from here that lower prevalence of the disease (i.e. lower $P(D+)$) would decrease $P(D+|T-)$ (the probability of a false negative).

You should be able to see that the prevalence of a disease, $P(D+)$, affects the PPV and NPV, but not the sensitivity or specificity, of a test. Hint: Use Bayes' rule.

3 Set Operations.

3.1 Part A

To answer this problem, first note that

$$\begin{aligned}
&P(B) = 0.1 \implies P(B^C) = 1 - 0.1 = 0.9 \\
&P(A^C|B) = 0.85 \implies P(A|B) = 1 - 0.85 = 0.15 \\
&P(A^C \cap B^C) = P(A^C|B^C)P(B^C) \\
&\quad \implies P(A^C|B^C) = P(A^C \cap B^C)/P(B^C) = \frac{0.3}{0.9} \\
&\quad \implies P(A|B^C) = 1 - \frac{0.3}{0.9} = \frac{2}{3}
\end{aligned}$$

(a)

$$\begin{aligned}
P(A \cap B^C) &= P(A|B^C)P(B^C) \\
&= (\frac{2}{3})(0.9) \\
&= 0.6
\end{aligned}$$

(b) Use law of total probability.

$$\begin{aligned}
P(A) &= P(A|B) + P(A|B^C) \\
&= 0.15 + \frac{2}{3} \\
&\approx 0.817
\end{aligned}$$

(c) Use Bayes' rule.

$$\begin{aligned}
P(B|A) &= \frac{P(A|B)P(B)}{P(A|B)P(B) + P(A|B^C)P(B^C)} \\
&= \frac{0.15 \cdot 0.1}{0.15 \cdot 0.1 + \frac{2}{3} \cdot 0.9} \\
&\approx 0.024
\end{aligned}$$

3.2 Part B

To answer this problem, first note that

$$\begin{aligned}
&P(B) = p \implies P(B^C) = 1 - p \\
&P(A^C|B) = q \implies P(A|B) = 1 - q \\
&P(A^C \cap B^C) = P(A^C|B^C)P(B^C) \\
&\quad \implies P(A^C|B^C) = P(A^C \cap B^C)/P(B^C) = \frac{r}{1-p} \\
&\quad \implies P(A|B^C) = 1 - \frac{r}{1-p}
\end{aligned}$$

(a)

$$\begin{aligned}
P(A \cap B^C) &= P(A|B^C)P(B^C) \\
&= (1 - \frac{r}{1-p})(1-p)
\end{aligned}$$

(b) Use law of total probability.

$$\begin{aligned} P(A) &= P(A|B) + P(A|B^C) \\ &= 1 - q + 1 - \frac{r}{1-p} \\ &= 2 - q + \frac{r}{1-p} \end{aligned}$$

(c) Use Bayes' rule.

$$\begin{aligned} P(B|A) &= \frac{P(A|B)P(B)}{P(A|B)P(B) + P(A|B^C)P(B^C)} \\ &= \frac{(1-q)p}{(1-q)p + (1-\frac{r}{1-p})(1-p)} \end{aligned}$$

4 Urns.

Let $W = \{\text{at least one white ball}\}$ and $D_i = \{\text{drew } i \text{ balls out}\}$. Then

$$P(D_i|W) = \frac{P(W|D_i)P(D_i)}{P(W)}.$$

Note that $P(W|D_i) = 1 - P(no\ white|D_i)$, that $P(D_i) = \frac{1}{4}\ \forall i$, and that $P(W) = 1 - P(no\ white) = \frac{1}{2}$. Thus

i	$P(no\ white\|D_i)$	$P(W\|D_i)$
1	$\frac{1}{2}$	$\frac{1}{2}$
2	$\frac{1}{2} \cdot \frac{1}{3} = \frac{1}{6}$	$\frac{5}{6}$
3	0	1
4	0	1

Thus we have

$$\begin{aligned} P(D_1|W) &= \frac{P(W|D_1)P(D_1)}{P(W)} \\ &= \frac{\frac{1}{2} \cdot \frac{1}{4}}{\frac{1}{2}} \\ &= \frac{1}{4} \end{aligned}$$

$$\begin{aligned} P(D_2|W) &= \frac{P(W|D_2)P(D_2)}{P(W)} \\ &= \frac{\frac{5}{6} \cdot \frac{1}{4}}{\frac{1}{2}} \\ &= \frac{5}{12} \end{aligned}$$

$$\begin{aligned} P(D_3|W) &= \frac{P(W|D_3)P(D_3)}{P(W)} \\ &= \frac{1 \cdot \frac{1}{4}}{\frac{1}{2}} \\ &= \frac{1}{2} \end{aligned}$$

$$
\begin{aligned}
P(D_4|W) &= \frac{P(W|D_4)P(D_4)}{P(W)} \\
&= \frac{1 \cdot \frac{1}{4}}{\frac{1}{2}} \\
&= \frac{1}{2}.
\end{aligned}
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