STAT310
Practice Problems
Week 3

February 6, 2012

1 Discrete random variables.

1. $X_1 \sim$ Negative-binomial(r=10, p=0.330) The negative binomial random variable counts the number of Bernoulli(p) trials required to get a fixed number of successes. In this case, the mean number of shots Lebron would have to take in order to make ten 3-pointers is equal to

$$\begin{aligned} E[X_1] &= r\frac{(1-p)}{p} \\ &= 10 \times \frac{0.67}{0.33} \\ &= 20.3. \end{aligned}$$

Also, we can see that the probability that he would make all ten 3-pointers that he attempted is equal to

$$\begin{aligned} P(X_1 = 10) &= \binom{10+10-1}{10}(0.33)^{10}(0.67)^{10} \\ &= (92378)(2.79e-07) \\ &= 0.0258. \end{aligned}$$

2. $X_2 \sim$ Bernoulli(0.330). The Bernoulli experiment is a single experiment with only two possible outcomes. In this case, the mean value of X_2 is

$$\begin{aligned} E[X_2] &= p \\ &= 0.330. \end{aligned}$$

Also, the probability that he will miss the shot is equal to

$$\begin{aligned} P(X_2 = 0) &= (0.330)^0(1-0.330)^{1-0} \\ &= 0.67. \end{aligned}$$

3. $X_3 \sim$ Binomial(4,0.33). Recall that a *binomial*(n,p) random variable is the sum of n *Bernoulli*(p) random variables. In this case, the mean value of X_3 is

$$\begin{aligned} E[X_3] &= np \\ &= (4)(0.33) \\ &= 1.32. \end{aligned}$$

The probability that he would make only one of these four shots, as he did in this particular game, is equal to

$$\begin{aligned} P(X_3 = 1) &= \binom{4}{1}(0.33)^1(0.67)^3 \\ &= (4)(0.099) \\ &= 0.397. \end{aligned}$$

4. $X_4 \sim$ discrete uniform(15). In this case, the mean number that I will pick is

$$\begin{aligned} E[X_4] &= \frac{15+1}{2} \\ &= 8 \end{aligned}$$

and the probability that I will pick Puppy #10 is equal to

$$P(X_4 = 10) = \frac{1}{15}.$$

5. $X_5 \sim$ Poisson($\lambda = \frac{5}{3}$). Recall that for the Poisson distribution, $E[X] = Var[X] = \lambda$! Thus the expected number of buses that pass by in the next minute is equal to

$$E[X_5] = Var[X_5] = \frac{5}{3}.$$

The probability of at least two buses in the next minute is equal to

$$\begin{aligned} P(X_5 \geq 2) &= 1 - P(X_5 = 0) - P(X_5 = 1) \\ &= 1 - 0.189 - \frac{e^{-5/3}\left(\frac{5}{3}\right)^1}{1!} \\ &= 0.496. \end{aligned}$$

2 Probability mass functions.

Recall that a function $f(x)$ is the **probability mass function (PMF)** of a random variable X if and only if (1) $f(x) \geq 0$ for all x, and (2) $\sum_{x \in S} f(x) = 1$. Note the "if and only if" part of the theorem! This ensures that any function that satisfies these two properties is the PMF of some random variable.

1. Yes, this is a PMF. We can see that $f(x) = \frac{x^2}{4} \geq 0$ for all $x = -1, 0, 1, \sqrt{2}$. Also,

$$\begin{aligned} \sum_{x \in S} \frac{x^2}{4} &= \frac{1}{4} + 0 + \frac{1}{4} + \frac{2}{4} \\ &= 1. \end{aligned}$$

The expected value of a random variable X with this PMF is given by

$$\begin{aligned} E[X] &= \sum_{x \in S} x \cdot \frac{x^2}{4} \\ &= \sum_{x} \frac{x^3}{4} \\ &= \frac{(-1)^3}{4} + \frac{0^3}{4} + \frac{1^3}{4} + \frac{\sqrt{2}^3}{4} \\ &= \frac{-1}{4} + \frac{1}{4} + \frac{\sqrt{2}^3}{4} \\ &= \frac{\sqrt{2}^3}{4} \end{aligned}$$

2. No, this is not a PMF. One can check that

$$\begin{aligned} \sum_{x \in S} f(x) &= \sum_{x \in S} e^x \\ &= e + e^2 + e^3 + e^4 + e^5 \\ &\neq 1. \end{aligned}$$

3. No, this is not a PMF. One can check that

$$\begin{aligned} \sum_{x \in S} f(x) &= \sum_{x \in S} \sqrt{x} \\ &= \sqrt{1} + \sqrt{3} + \sqrt{5} + \sqrt{9} + \sqrt{12}. \end{aligned}$$

4. Yes, this is a PMF. We can see that $f(x) = \frac{x^4}{10} \geq 0$ for all $x = -4^{1/4}, -1, 1, 4^{1/4}$. Also,

$$\begin{aligned} \sum_{x \in S} \frac{x^4}{10} &= \frac{4}{10} + \frac{1}{10} + \frac{1}{10} + \frac{4}{10} \\ &= 1. \end{aligned}$$

The expected value of a random variable X with this PMF is given by

$$\begin{aligned} E[X] &= \sum_{x \in S} x \cdot \frac{x^4}{10} \\ &= \sum_{x \in S} \frac{x^5}{10} \\ &= \frac{(-4^{1/4})^5}{10} + \frac{(-1)^5}{10} + \frac{1^5}{10} + \frac{(4^{1/4})^5}{10} \\ &= 0 \end{aligned}$$

3 Recognizing PMF's.

1. $A \sim Binomial(3, c)$

2. $C \sim Poisson(-(a+b))$

3. $B \sim Bernoulli(a)$. To see this, recall that

$$\left(\sum_{k=0}^{\infty} x^k\right)^p = (1-x)^{-p},$$

so that

$$\begin{aligned} (1-a)^{1-b} &= (1-a)^{-(b-1)} \\ &= \left(\sum_{k=0}^{\infty} a^k\right)^{b-1} \end{aligned}$$

4. $X \sim discrete - uniform(1)$. Note that this is a PMF that puts weight 1 at a single point $x = 1$ only.