STAT310
Practice Problems
Week 4

February 9, 2012

1 Moment generating functions (I).

1.

$$\begin{aligned} E[e^{tX}] &= \int_0^c e^{tx}\frac{1}{c}dx \\ &= \frac{1}{ct}e^{tx}\rfloor_0^c \\ &= \frac{1}{ct}e^{tc} - \frac{1}{ct}1 \\ &= \frac{1}{ct}(e^{tc}-1) \end{aligned}$$

2.

$$\begin{aligned} E[e^{tX}] &= \int_0^c \frac{2x}{c^2}e^{tx}dx \\ &= \frac{2}{c^2t^2}(cte^{tc} - e^{tc} + 1) \;\; (\textit{integration by parts}) \end{aligned}$$

3.

$$\begin{aligned} E[e^{tx}] &= \int_{-\infty}^{\alpha} \frac{1}{2\beta}e^{(x-\alpha)/\beta}e^{tx}dx + \int_{\alpha}^{\infty} \frac{1}{2\beta}e^{-(x-\alpha)/\beta}e^{tx}dx \\ &= \frac{e^{-\alpha/\beta}}{2\beta}\frac{1}{(\frac{1}{\beta}+t)}e^{x(\frac{1}{\beta}+t)}\rfloor_{-\infty}^{\alpha} + -\frac{e^{\alpha/\beta}}{2\beta}\frac{1}{(\frac{1}{\beta}-t)}e^{-x(\frac{1}{\beta}-t)}\rfloor_{\alpha}^{\infty} \\ &= \frac{4e^{\alpha t}}{4-\beta^2t^2}, \quad -2/\beta < t < 2/\beta \end{aligned}$$

2 Moment generating functions (II).

1. $X \sim$ Discrete uniform (N).
2. $X \sim$ Bernoulli(p).
3. $X \sim$ Binomial(n,p).

4. $X \sim$ Poisson(λ).

5. $X \sim$ Negative binomial(r,p).

6. $X \sim$ Geometric(p).

3 Moment generating functions (III).

1.

$$\begin{aligned}
M_X(t) = E[e^{tX}] &= \int_{-\infty}^{0} \frac{1}{2} e^{-|x|} e^{tx} dx \\
&= \int_{-\infty}^{0} \frac{1}{2} e^{x} e^{tx} dx \\
&= \frac{1}{2} \int_{-\infty}^{0} e^{x(1+t)} dx \\
&= \frac{1}{2} \left[\frac{1}{1+t} e^{x(1+t)} \right]_{-\infty}^{0} \\
&= \frac{1}{2(1+t)}
\end{aligned}$$

$$\begin{aligned}
E[X] &= M_X^{(1)}(0) \\
&= \frac{\partial}{\partial t} \left[\frac{1}{2} (1+t)^{-1} \right]_{t=0} \\
&= \left[-\frac{1}{2} (1+t)^{2} \right]_{t=0} \\
&= -\frac{1}{2}
\end{aligned}$$

$$\begin{aligned}
E[X^2] &= M_X^{(2)}(0) \\
&= \frac{\partial}{\partial t} \left[-\frac{1}{2} (1+t)^{2} \right]_{t=0} \\
&= \left[(1+t)^{-3} \right]_{t=0} \\
&= 1
\end{aligned}$$

$$\begin{aligned}
Var[X] &= E[X^2] - E[X]^2 \\
&= 1 - (-\frac{1}{2})^2 \\
&= \frac{3}{4}
\end{aligned}$$

2.

$$
\begin{aligned}
M_X(t) = E[e^{tX}] &= \int_0^\infty e^{tx} c e^{-x} dx \\
&= c \int_0^\infty e^{-x(1-t)} dx \\
&= \frac{c}{t-1} \left[e^{-x(1-t)} \right]_0^\infty \\
&= \frac{c}{t-1} [0-1] \\
&= \frac{c}{1-t}
\end{aligned}
$$

$$
\begin{aligned}
E[X] &= M_X^{(1)}(0) \\
&= \frac{\partial}{\partial t} \left[c(1-t)^{-1} \right]_{t=0} \\
&= \left[c(1-t)^{-2} \right]_{t=0} \\
&= c
\end{aligned}
$$

$$
\begin{aligned}
E[X^2] &= M_X^{(2)}(0) \\
&= \frac{\partial}{\partial t} \left[c(1-t)^{-2} \right]_{t=0} \\
&= \left[2c(2-t)^{-3} \right]_{t=0} \\
&= 2c
\end{aligned}
$$

$$
\begin{aligned}
Var[X] &= E[X^2] - E[X]^2 \\
&= 2c - c^2
\end{aligned}
$$

4 Continuous random variables.

Recall that a function is a PDF iff (1) $\int_\infty^\infty f(x)dx = 1$, and (2) $f(x) \geq 0 \ \forall x \in S$, where S denotes the sample space of X.

1. No, $f(x) = x^3, \ x \in (-10, 10)$ is not a PDF. One can check that

$$
\begin{aligned}
\int_{-10}^{10} x^3 dx &= \left[\frac{x^4}{4} \right]_{-10}^{10} \\
&= \frac{10^4}{4} - \frac{(-10)^4}{4} \\
&= 0.
\end{aligned}
$$

2. No, $f(x) = x^3, \ x \in (0, 10)$ is not a PDF. One can check that

$$
\begin{aligned}
\int_0^\infty x^3 dx &= \left[\frac{x^4}{4} \right]_0^{10} \\
&= \frac{10^4}{4} \\
&> 1.
\end{aligned}
$$

3. Yes, $f(x) = \frac{x^2}{4}$, $x \in (-6^{1/3}, 6^{1/3})$ is a PDF. One can check that

$$\begin{aligned}
\int_{-6^{1/3}}^{6^{1/3}} \frac{x^2}{4} dx &= \left[\frac{1}{12}x^3\right]_{-6^{1/3}}^{6^{1/3}} \\
&= \frac{1}{12}[6+6] \\
&= 1.
\end{aligned}$$

Additionally, it is easy to verify that $\frac{x^2}{4} \geq 0 \; \forall x \in \mathbb{R}$.

(a) Using the PDF, we can calculate the desired probabilities as

$$\begin{aligned}
P(X > 0) &= \int_0^{6^{1/3}} \frac{x^2}{4} dx \\
&= \left[\frac{x^3}{12}\right]_0^{6^{1/3}} \\
&= \frac{6}{12} \\
&= \frac{1}{2}
\end{aligned}$$

$$\begin{aligned}
P(1 < X \leq 2) &= \int_1^2 \frac{x^2}{4} dx \\
&= \left[\frac{x^3}{12}\right]_1^2 \\
&= \frac{8}{12} - \frac{1}{12} \\
&= \frac{7}{12}
\end{aligned}$$

(b) To calculate the CDF, we have that

$$\begin{aligned}
F_X(x) &= \int_{-6^{1/3}}^{x} \frac{x^2}{4} dx \\
&= \left[\frac{x^3}{12}\right]_{-6^{1/3}}^{x} \\
&= \frac{x^3}{12} + \frac{6}{12} \\
&= \frac{x^3+6}{12}
\end{aligned}$$

(c) Using the CDF found above, we can calculate the desired probabilities as

$$\begin{aligned}
P(X > 0) &= 1 - F_X(0) \\
&= 1 - \frac{6}{12} \\
&= \frac{1}{2},
\end{aligned}$$

which is the same as in Part A. Also,

$$\begin{aligned}
P(1 < X \leq 2) &= F_X(2) - F_X(1) \\
&= \frac{2^3+6}{12} - \frac{1^3+6}{12} \\
&= \frac{14}{12} - \frac{7}{12} \\
&= \frac{7}{12},
\end{aligned}$$

which is the same as in Part A.

4. Yes, $f(x) = \frac{3}{16}\sqrt{x},\ x \in [0,4]$ is a PDF. One can check that

$$\begin{aligned}
\int_0^4 \frac{3}{16} x^{1/2} dx &= \left[\frac{3}{16}\frac{2}{3}x^{3/2}\right]_0^4 \\
&= \frac{1}{8}[8-0] \\
&= 1.
\end{aligned}$$

Additionally, one can check that $f(x) = \frac{3}{16}\sqrt{x} \geq 0\ \forall x \in [0,4]$, since $\frac{3}{16} > 0$ and $\sqrt{x} \geq 0\ \forall x \in [0,4]$.

(a) Using the PDF, we can calculate the desired probabilities as

$$\begin{aligned}
P(X > 0) &= \int_0^4 \frac{3}{16}\sqrt{x} dx \\
&= \left[\frac{1}{8}x^{3/2}\right]_0^4 \\
&= 1.
\end{aligned}$$

Intuitively, this makes sense because the support of X is defined on $[0,4]$. We can also calculate

$$\begin{aligned}
P(1 < X \leq 2) &= \int_1^2 \frac{3}{16}\sqrt{x} dx \\
&= \left[\frac{1}{8}x^{3/2}\right]_1^2 \\
&= \frac{8^{1/2}}{8} - \frac{1}{8} \\
&\approx 0.229.
\end{aligned}$$

(b) To calculate the CDF, we have that

$$\begin{aligned}
F_X(x) &= \int_0^x \frac{3}{16}\sqrt{x} dx \\
&= \left[\frac{1}{8}x^{3/2}\right]_0^x \\
&= \frac{1}{8}x^{3/2}.
\end{aligned}$$

(c) Using the CDF found above, we can calculate the desired probabilities as

$$\begin{aligned} P(X > 0) &= 1 - F_X(0) \\ &= 1 - \frac{1}{8}0^{3/2} \\ &= 1 - 0 \\ &= 1, \end{aligned}$$

which is the same as in Part A. Also,

$$\begin{aligned} P(1 < X \leq 2) &= F_X(2) - F_X(1) \\ &= \frac{1}{8}2^{3/2} - \frac{1}{8} \\ &\approx 0.229. \end{aligned}$$

which is also the same as in Part A.