STAT310
Practice Problems
Week 5

March 26, 2012

1 Bivariate PDFs.

1. Yes, this is a PDF. We have that

$$\begin{aligned}
\int_{[0,1]}\int_{[0,1]} f(x,y) &= \int_0^1\int_0^1 2x dx dy \\
&= \int_0^1 \left[x^2\right]_0^1 dy \\
&= \int_0^1 dy \\
&= [y]_0^1 \\
&= 1.
\end{aligned}$$

   Also, $f(x,y) \geq 0 \ \forall x \in [0,1], \ y \in [0,1]$.

2. Yes, this is a PDF. Note that

$$\begin{aligned}
\int_{[0,1]}\int_{[0,1]} f(x,y) &= \int_0^1\int_0^1 (x+y) dx dy \\
&= \int_0^1 \left[\frac{x^2}{2} + yx\right]_0^1 dy \\
&= \int_0^1 \left[\frac{1}{2} + y\right] dy \\
&= \left[\frac{1}{2}y + \frac{y^2}{2}\right]_0^1 \\
&= \frac{1}{2} + \frac{1}{2} \\
&= 1.
\end{aligned}$$

   Also, $f(x,y) \geq 0 \ \forall x \in [0,1], \ y \in [0,1]$.

3. No, this is not a PDF, since

$$
\begin{aligned}
\int_{[0,1]} \int_{[0,1]} f(x,y) &= \int_0^1 \int 0^1 (x^3 + y/2) dx dy \\
&= \int_0^1 \left[\frac{x^4}{4} + \frac{y}{2} x \right]_0^1 dy \\
&= \int_0^1 \left(\frac{1}{4} + \frac{y}{2} \right) dy \\
&= \left[\frac{1}{4} y + \frac{y^2}{4} \right]_0^1 \\
&= \frac{1}{4} + \frac{1}{4} \\
&= \frac{1}{2}
\end{aligned}
$$

2 Independent random variables.

1. No, X and Y are not independent because $f(x,y) \neq f_X(x) f_Y(y) \quad \forall x, y$. We can easily calculate the marginal PMFs are

$$
\begin{aligned}
&f_X(10) = f_X(20) = \frac{1}{2} \\
&f_Y(1) = \frac{1}{5} \\
&f_Y(2) = \frac{3}{10} \\
&f_Y(3) = \frac{1}{2}.
\end{aligned}
$$

Then we can see that X and Y are not independent, since for example

$$f(10,3) = \frac{1}{5} \neq \frac{1}{2} \cdot \frac{1}{2} = f_X(10) f_Y(3).$$

2. Define

$$
\begin{aligned}
g(x) &= x^2 e^{-x/2} \\
h(y) &= y^4 e^{-y}/384
\end{aligned}
$$

Then $f(x,y) = g(x)h(y)$. Thus X and Y are independent random variables.

3. Yes, X and Y are independent random variables. Define

$$
\begin{aligned}
&g(x) = x \\
&h(y) = y + y^3.
\end{aligned}
$$

Then $f(x,y) = g(x)h(y)$.

4. Yes, X and Y are independent random variables. Define

$$
\begin{aligned}
&g(x) = x^2 e^x \\
&h(y) = e^y.
\end{aligned}
$$

Then $f(x,y) = g(x)h(y)$.

3 Conditional distributions.

1. If $x \leq 0$, then $f(x,y) = 0 \ \ \forall y$, so $f_X(x) = 0 \forall x \leq 0$. If $x > 0$, then $f(x,y) > 0$ only if $y > x$. Thus

$$\begin{aligned} f_X(x) &= \int_{-\infty} \infty f(x,y) dy \\ &= \int_x^{\infty} e^{-y} dy \\ &= e^{-x}. \end{aligned}$$

2. Note that we can calculate $f(y|x)$ $\forall x > 0$, since these are the values for which $f_X(x) > 0$. We have

$$f(y|x) = \frac{f(x,y)}{f_X(x)} = \frac{e^{-y}}{e^{-x}} = e^{-(y-x)}, \quad if \ y > x$$
$$f(y|x) = \frac{f(x,y)}{f_X(x)} = \frac{0}{e^{-x}} = 0, \quad if \ y \leq x.$$

3. We have

$$\begin{aligned} f(x,y) &= \left(\frac{x+y}{3y+6} \right) \left(\frac{3y+6}{21} \right) \\ &= \frac{x+y}{21} \end{aligned}$$

$\forall x = 1, 2, 3, \ y = 1, 2.$