STAT310
Practice Problems
Week 8

March 26, 2012

1 Bivariate change of variables.

1. The joint PMF of (X, Y) is

$$f_{X,Y}(x,y) = \frac{\theta e^{-\theta}}{x!}\frac{\lambda^y e^{-\lambda}}{y!}, \quad x = 0,1,2,..., \ y = 0,1,2,...$$

The range space $\mathcal{A}$ of (X,Y) is $\mathcal{A} = \{(x,y) : x = 0,1,2,... \ \mathit{and} \ y = 0,1,2,...\}$. Defining $U = X + Y$ and $V = Y$. Then the range space $\mathcal{B}$ of (U,V) is $\mathcal{B} = \{(u,v) : v = 0,1,2,... \ \mathit{and} \ u = v, v+1, v+2, ...\}$. For any $(u,v) \in \mathcal{B}$, the only (x,y) value satisfying $x + y = u$ and $y = v$ is $x = u - v$ and $y = v$. Thus

$$f_{UV}(u,v) = f_{XY}(u-v,v) = \frac{\theta^{u-v}e^{-\theta}}{(u-v)!}\frac{\lambda^v e^{-\lambda}}{v!},$$

$v = 0,1,2,...,\ u = v, v+1, v+2, ...$ Thus

$$\begin{aligned} f_U(u) &= \sum_{v=0}^{u} \frac{\theta^{u-v}e^{-\theta}}{(u-v)!}\frac{\lambda^v e^{-\lambda}}{v!} \\ &= e^{-(\theta+\lambda)} \sum_{v=0}^{u} \frac{\theta^{u-v}}{(u-v)!}\frac{\lambda^v}{v!} \\ &= \frac{e^{-(\theta+\lambda)}}{u!} \sum_{v=0}^{u} \binom{u}{v} \lambda^v \theta^{u-v} \\ &= \frac{e^{-(\theta+\lambda)}}{u!}(\theta+\lambda)^u, \end{aligned}$$

$u = 0,1,2,...,$ which is the PMF of a Poisson r.v. with parameter $\theta + \lambda$. In other words, if $X \sim \mathit{Poisson}(\theta)$ and $Y \sim \mathit{Poisson}(\lambda)$ and X and Y are independent, then $X + Y \sim \mathit{Poisson}(\theta + \lambda)$.

2. The joint PDF of (X,Y) is

$$f_{XY}(x,y) = \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)}x^{\alpha-1}(1-x)^{\beta-1}\frac{\Gamma(\alpha+\beta+\gamma)}{\Gamma(\alpha+\beta)\Gamma(\gamma)}yu^{\alpha+\beta-1}(1-y)^{\gamma-1}.$$

Consider the transformation $U = XY$ and $V = X$. The range space for V is $(0,1)$ since $V = X$. For a given $V = v$, $u \in (0, v)$ since $X = V = v$ and Y is between 0 and 1. Thus the range space of (U,V)

is $\mathcal{B} = \{(u,v) : 0 < u < v < 1\}$, and the range space of (X,Y) is $\mathcal{A} = \{(x,y) : x \in (0,1),\ y \in (0,1)\}$. Note that this is a one-to-one transformation of $\mathcal{A}$ onto $\mathcal{B}$. The Jacobian is given by

$$\begin{array}{ccc} to & & \\ J & = & \| \begin{pmatrix} 0 & 1 \\ \frac{1}{v} & -\frac{u}{v^2} \end{pmatrix} \| \end{array}$$

so

$$f_{UV}(u,v) = \frac{\Gamma(\alpha+\beta+\gamma)}{\Gamma(\alpha)\Gamma(\beta)\Gamma(\gamma)} v^{\alpha-1}(1-v)^{\beta-1}\left(\frac{u}{v}\right)^{\alpha+\beta-1}\left(1-\frac{u}{v}\right)\gamma - 1\frac{1}{v},$$

$0 < u < v < 1$. The marginal distribution of U is

$$\begin{aligned} f_U(u) &= \int_u^1 f_{U,V}(u,v)dv \\ &= \frac{\Gamma(\alpha+\beta+\gamma)}{\Gamma(\alpha)\Gamma(\beta)\Gamma(\gamma)} u^{\alpha-1}\int_u^1 \left(\frac{u}{v}-u\right)^{\beta-1}\left(1-\frac{u}{v}\right)^{\gamma-1}\left(\frac{u}{v^2}\right)dv \\ &= \frac{\Gamma(\alpha+\beta+\gamma)}{\Gamma(\alpha)\Gamma(\beta)\Gamma(\gamma)} u^{\alpha-1}(1-u)^{\beta+\gamma-1}\int_0^1 y^{\beta-1}(1-y)^{\gamma-1}dy \quad (set\ y = (u/v-u)/(1-u)) \\ &= \frac{\Gamma(\alpha+\beta+\gamma)}{\Gamma(\alpha)\Gamma(\beta)\Gamma(\gamma)} u^{\alpha-1}(1-u)^{\beta+\gamma-1}\frac{\Gamma(\beta)\Gamma(\gamma)}{\Gamma(\beta+\gamma)} \\ &= \frac{\Gamma(\alpha+\beta+\gamma)}{\Gamma(\alpha)\Gamma(\beta+\gamma)} u^{\alpha-1}(1-u)^{\beta+\gamma-1}, \end{aligned}$$

$0 < u < 1$, so that $U \sim beta(\alpha, \beta+\gamma)$.

3. Note that this transformation is not one-to-one since the points (x,y) and $(-x,-y)$ are both mapped into the same (u,v) point. We must partition the range space $\mathcal{A} = \mathbb{R}^2$ of (X,Y), i.e. let

$$\begin{aligned} A_1 &= \{(x,y) : y > 0\} \\ A_2 &= \{(x,y) : y < 0\} \\ A_0 &= \{(x,y) : y = 0\}, \end{aligned}$$

so that A_0, A_1, and A_2 form a partition of $\mathcal{A} = \mathbb{R}^2$ and $P((X,Y) \in A_0) = P(Y =)) = 0$. Then the range space $\mathcal{B}$ of (U,V) is $\mathcal{B} = \{(u,v) : v > 0\}$. The inverse transformations from $\mathcal{B}$ to A_1 and *mathcalB* to A_2 are given by $x = h_{11}(u,v) = uv$, $y = h_{21}(u,v) = v$, $x = h_{12}(u,v) = -uv$, and $y = h_{22}(u,v) = -v$. The Jacobians from the two inverses are $J_1 = J_2 = v$. Using

$$f_{XY}(x,y) = \frac{1}{2\pi}e^{-x^2/2}e^{-y^2/2},$$

we have

$$\begin{aligned} f_{UV}(u,v) &= \frac{1}{2\pi}e^{-(uv)^2/2}e^{-v^2/2}|v| + \frac{1}{2\pi}e^{-(-uv)^2/2}e^{-(-v)^2/2}|v| \\ &= \frac{v}{\pi}e^{-(u^2+1)v^2/2}, \end{aligned}$$

$-\infty < u < \infty,\ 0 < v < \infty$. Thus

$$\begin{aligned}
f_U(u) &= \int_0^\infty \frac{v}{\pi} e^{-(u^2+1)v^2/2} dv \\
&= \frac{1}{2\pi} \int_0 e^{-(u^2+1)z/2} dz \quad (let\ z = v^2) \\
&= \frac{1}{2\pi} \frac{2}{(u^2+1)} \quad (integrand\ is\ kernel\ of\ expo(\beta = 2/(u^2+1)) \\
&= \frac{1}{\pi(u^2+1)},
\end{aligned}$$

$-\infty < u < \infty$. Thus the ratio of two independent standard normal r.v.'s is Cauchy.