STAT310
Practice Problems
Week 9

March 26, 2012

1 Limiting distributions.

1.

$$
\begin{aligned}
& X_n \sim \mathrm{N}(\frac{n+1}{n}, \sigma^2) \\
\Longrightarrow \quad & X_n \sim \mathrm{N}(1+\frac{1}{n}, \sigma^2) \\
\Longrightarrow \quad & \lim_{n\to\infty} X_n \sim \mathrm{N}(\lim_{n\to\infty}(1+\frac{1}{n}), \sigma^2) \\
\Longrightarrow \quad & \lim_{n\to\infty} X_n \sim \mathrm{N}(1, \sigma^2)
\end{aligned}
$$

2. The probability density function of Y_n is given by $f_{\alpha,\beta}(y_n) = \frac{1}{\Gamma(1+e^{-n})\beta^{1+e^{-n}}} x^{1+e^{-n}-1} e^{-x/\beta}$. Thus

$$
\begin{aligned}
& \lim_{n\to\infty} \frac{1}{\Gamma(1+e^{-n})\beta^{1+e^{-n}}} x^{1+e^{-n}-1} e^{-x/\beta} \\
= \quad & \frac{1}{\Gamma(1)\beta} e^{-x/\beta} \\
= \quad & \frac{1}{\beta} e^{-x/\beta}
\end{aligned}
$$

which is the probability density function of an Exponential(β) r.v.

3. As $n \to \infty$, $Var(Z_n) \to 0$. Thus the limiting distribution of Z_n is a "spike" at 1, where the probability density function is given by

$$
\delta_1(z) = \left\{ \begin{array}{c} 1, \ z = 1 \\ 0, \ otherwise \end{array} \right\}
$$

2 Law of large numbers and Chebyshev's inequality.

The law of large numbers say that the sample mean $\bar{X}_n$ converges in probability to the mean μ. Using Chebyshev's inequality,

$$
\begin{aligned}
& P(|\bar{X}_n - \mu| \geq \frac{k\sigma}{\sqrt{n}}) \leq \frac{1}{k^2} = 0.9 \\
& 1 - P(|\bar{X}_n - \mu| \geq \frac{k\sigma}{\sqrt{n}}) \geq 1 - \frac{1}{k^2} = 0.9 \\
& P(|\bar{X}_n - \mu| \leq \frac{k\sigma}{\sqrt{n}} = \frac{\sigma}{2}) \geq 1 - \frac{1}{k^2} = 0.9 \\
\implies & \frac{1}{k^2} = 0.1 \\
\implies & k = \sqrt{10} \\
\implies & \frac{k}{\sqrt{n}} = \frac{1}{2} \\
\implies & \frac{\sqrt{10}}{\sqrt{n}} = \frac{1}{2} \\
\implies & \frac{10}{n} = \frac{1}{4} \\
\implies & n = 40
\end{aligned}
$$

3 Central limit theorem.

1. If we were to find $P(\bar{X} \leq 11)$ using an exact calculation, we would have

$$
\begin{aligned}
P(\bar{X} \leq 11) &= P\left(\sum_{i=1}^{30} X_i \leq 330\right) \\
&= \sum_{x=0}^{330} \binom{300 + x - 1}{x} \left(\frac{1}{2}\right)^{300} \left(\frac{1}{2}\right)^{x}
\end{aligned}
$$

   which is very difficult to calculate, even using Wolfram Alpha. On the other hand, the CLT approximation gives us

$$
\begin{aligned}
P(\bar{X} \leq 11) &= P\left(\frac{\sqrt{30}(\bar{X} - 10)}{\sqrt{20}} \leq \frac{\sqrt{30}(11 - 10)}{\sqrt{20}}\right) \\
&\approx P(X \leq 1.2247) = 0.8888.
\end{aligned}
$$

2. Using the CLT, $\bar{X}$ is approximately $\mathrm{N}(\mu, \sigma^2_{\bar{X}})$ with $\sigma_{\bar{X}} = \sqrt{0.09} = 0.3$ and $(\bar{X} - \mu)/0.3 \sim \mathrm{N}(0, 1)$. Thus

$$
\begin{aligned}
0.9 &= P\left(-1.645 < \frac{\bar{X} - \mu}{0.3} < 1.645\right) \\
&= P(-0.4935 < \bar{X} - \mu < 0.4935).
\end{aligned}
$$

4 Moment generating functions.

1. Since X_i are independent and identically distributed, then the MGF of Y is given by

$$\begin{aligned} \psi_Y(t) &= \psi_{X_1+X_2+...+X_n}(t) \\ &= \psi_{X_1}(t)\psi_{X_2}(t)...\psi_{X_n}(t) \\ &= [pe^t + (1-p)]^n \end{aligned}$$

   since the MGF of each Bernoulli r.v. X_i is given by $pe^t + (1-p)$.

2. Since X_i are independent, then the MGF of Y is given by

$$\begin{aligned} \psi_Y(t) &= \psi_{X_1+X_2+...+X_n}(t) \\ &= \psi_{X_1}(t)\psi_{X_2}(t)...\psi_{X_n}(t) \\ &= \prod_{i=1}^{n} \frac{e^{ta_i}-1}{ta_i} \end{aligned}$$

   which does not correpond to a known distribution.