STAT310
Practice Problems
Week 10

April 9, 2012

# 1 Maximum likelihood estimators.

1. The likelihood function is

$$\begin{aligned} L(\beta|x) &= \prod_{i=1}^{n} \frac{1}{\Gamma(\alpha)\beta^{\alpha}} x_i^{\alpha-1} e^{-x_i/\beta} \\ &= \frac{1}{\Gamma(\alpha)^n \beta^{n\alpha}} \left[\prod_{i=1}^{n} x_i\right]^{\alpha-1} e^{-\sum_{i=1}^{n} x_i/\beta}. \end{aligned}$$

Thus the log-likelihood is

$$logL(\beta|x) = -log\Gamma(\alpha)^n - n\alpha log\beta + (\alpha-1)log\left[\prod_{i=1}^{n} x_i\right] - \frac{\sum_{i=1}^{n} x_i}{\beta}$$

and the first derivative of the log-likelihood is

$$\frac{\partial logL}{\partial \beta} = -\frac{n\alpha}{\beta} + \frac{\sum_{i=1}^{n} x_i}{\beta^2}$$

Setting $\frac{\partial logL}{\partial \beta} = 0$ and solving for $\beta$ yields $\hat{\beta}_{MLE} = \frac{sum_{i=1}^{n} x_i}{n\alpha}$. To check this is a maximum:

$$\begin{aligned} \frac{\partial^2 logL}{\partial \beta^2}\Big|_{\beta=\hat{\beta}_{MLE}} &= \frac{n\alpha}{\beta^2} - \frac{2\sum_{i=1}^{n} x_i}{\beta^3}\Big|_{\beta=\hat{\beta}} \\ &= \frac{(n\alpha)^3}{(\sum_{i=1}^{n} x_i)^2} - \frac{2(n\alpha)^3}{(\sum_{i=1}^{n} x_i)^2} \\ &= -\frac{(n\alpha)^3}{(\sum_{i=1}^{n} x_i)^2} < 0 \end{aligned}$$

2. The likelihood function is

$$\begin{aligned} L(\theta|x) &= \prod_{i=1}^{n} \frac{1}{\theta} I_{0,\theta}(x_i) \\ &= \frac{1}{\theta^n} I_{[0,\theta]}(x_{(n)}) I_{[0,\infty]}(x_{(1)}) \end{aligned}$$

where $x_{(1)}$ and $x_{(n)}$ are the minimum and maximum order statistics, respectively. For $\theta \geq x_{(n)}$, $L = \frac{1}{\theta_n}$, which is a decreasing function. Thus for $\theta \geq x_{(n)}$, $L(\theta|x)$ is maximized at $\hat{\theta} = x_{(n)}$. For $\theta < x_{(n)}$, $L(\theta|x) = 0$. Thus the MLE is $\hat{\theta}_{MLE} = X_{(n)}$.

3. The likelihood function is

$$\begin{aligned} L(p|x) &= (1-p)p^{x_1}(1-p)p^{x_2}...(1-p)p^{x_n} \\ &= (1-p)^n p^{\sum x_i}. \end{aligned}$$

The log-likelihood is then

$$logL(p|x) = nlog(1-p) + \sum_i x_i log(p)$$

and

$$\frac{\partial logL}{\partial p} = -\frac{n}{1-p} + \frac{\sum x_i}{p}.$$

Setting the first derivative equal to 0 and solving for $p$, we get

$$\begin{aligned} &\frac{\sum x_i}{p} = \frac{n}{1-p} \\ &\implies np = (1-p)\sum x_i \\ &\implies np = \sum x_i - p\sum x_i \\ &\implies p(n + \sum x_i) = \sum x_i \\ &\implies p = \frac{\sum x_i}{n + \sum x_i}. \end{aligned}$$

Thus $\hat{p}_{MLE} = \frac{\sum x_i}{n+\sum x_i}$.

4. We found in Question 2 that$\hat{\theta}_{MLE} = X_{(n)}$. The PDF for $\hat{\theta} = X_{(n)}$ is $\frac{nx^{n-1}}{\theta^n}$ $\forall x \in [0, \theta]$. Thus

$$E[\hat{\theta}_{MLE}] = \frac{n}{n+1}\theta.$$