Stat310: Test 1

Answer the questions in the spaces provided on the question sheets. If you run out of room for an answer, continue on the back of the page.
You have **two hours** and you may use one double-sided page of notes, and wolframalpha if you have to. You may not use your text book, or communicate anything about the contents of this test to anyone else. Please pledge on the last page.

Name: ______________________________

1. On average, I call on three people at random in each class, and over the course of the semester we'll have 28 classes in total. There are 140 students in stat310.

   (a) What's the probability that I call on you two or more times during the semester? (4)

Let X = number of times I call on you

$X \sim \text{Binomial}(n = 3 \times 28 = 84,\ p = \frac{1}{141})$ 1 pt convert to mathematics!

1 pt complement

$P(X \geq 2) = 1 - P(X < 2)$

$= 1 - P(X = 1) - P(X = 0)$

$= 1 - \binom{140}{1} \frac{1}{140} \left(\frac{139}{140}\right)^{83} - \binom{140}{0}\left(\frac{1}{140}\right)^0 \frac{139}{140}^{84}$ 1 pt intermediate steps.

$= 1 - 0.371 + 0.548$

$= 0.121$ 1 pt correct answer

(b) What's the probability that **someone** in the class gets called on more than once in a semester. (Hint: think about the complement) (2)

Prob nobody gets called more than once = every call is unique: $\frac{140}{140} \cdot \frac{139}{140} \cdots \frac{56}{140} = \frac{140!/55!}{140^{84}} \simeq 5 \times 10^{-13}$

2pt for reasonably close to this reasoning

$\Rightarrow$ P(someone gets called on more than once) $\cong 1$.

(c) Should your answer to part (b) be big (close to 1) or small (close to 0)? Why? (2)

Big 1 pt

* already happened to two people

* 12% each for 140 people = big chance

* ...

– any good reason = 1 pt

2. A helpful mathematical fact: $\sum_{i=0}^{\infty} x^i/i! = e^x$

(a) Verify that the pmf of the Poisson distribution, $f(x) = e^{-\lambda}\lambda^x/x!$ is a true pmf. (3)

$f(x) \geq 0 \Rightarrow e^{-\lambda} \geq 0,\ \lambda^x \geq 0,\ x! \geq 0$ 1 pt

$\sum f(x) = 1 \Rightarrow \sum_{x=0}^{\infty} \frac{e^{-\lambda}\lambda^x}{x!} = e^{-\lambda} \sum_{x=0}^{\infty} \frac{\lambda^x}{x!} = e^{-\lambda} e^{\lambda} = 1 \quad \square$

1 pt 1 pt

(b) Complete the blanks to find the mean of the poisson distribution. (7)

$X \sim \text{Poisson}(\lambda)$

$E(X) = \underline{\sum_{x \in S} x\, f(x)}$ 1 pt by definition of $E(X)$

$= \underline{\sum_{x=0}^{\infty} x \frac{e^{-\lambda} \lambda^x}{x!}}$ 1 pt by definition of X

$= \sum_{x=1}^{\infty} x \frac{e^{-\lambda} \lambda^x}{x!}$ first term $= 0$ 1 pt

$= e^{-\lambda} \lambda \underline{\sum_{x=1}^{\infty} \frac{\lambda^{x-1}}{(x-1)!}}$ 1 pt moving out constants

$= \underline{e^{-\lambda} \lambda \sum_{y=0}^{\infty} \frac{\lambda^y}{y!}}$ 1 pt $y = x - 1$

$= \underline{e^{-\lambda} \lambda\, e^{\lambda}}$ 1 pt random mathematical fact 1 pt

$= \lambda$ $\square$

3. On average it takes me 10 minutes to drive to Rice from home.

(a) Give three characteristics that the distribution of my driving times might reasonably possess. (3)

* continuous
* skew right
* non-negative
* bounded either side ...

1pt each for any reasonable characteristic

Assume that my driving times, D, has a exponential distribution. (This is not a very appropriate distribution, but it makes the rest of the question possible)

(b) If I leave home at 12:45pm, what are my chances of making it to class by 1? (2)

$$P(D \leq 15) = F_D(15) - F_D(0)$$

$$= 1 - e^{-15/10} - (\cancel{1 - e^{-0/10}})$$

$$= 1 - e^{-1.5} = 0.7765$$

1pt problem set up

1pt correct answer

(c) What time do I need to leave home to have a 99% chance of making it to class on time? (2)

$$P(D \leq d) = 0.99$$

$$1 - e^{-d/10} = 0.99$$

$$0.01 = e^{-d/10}$$

$$10 \times \ln(0.01) = -d$$

$$d = -10 \ln 0.01 \approx 46 \text{ min}$$

$$\Rightarrow 12\,14 \text{ pm}$$

1 pt set up

1pt answer.

(d) Instead of looking at the distribution of time, I could look at the distribution of my velocity. It's 3.6 miles from my home to Rice. What's the distribution of my speed in miles per hour? Work it out by transforming the distribution of time. (5)

+1 point for $y = \frac{216}{x}$

Let Y be the speed I travel. Then (in miles per hour) $y = \frac{3.6 \text{ miles}}{X \text{ minutes}} \cdot 60 = \frac{216}{X}$ mph

To find the distribution of y, we use the distribution function technique:

$y = \frac{216}{x} \quad x \in [0,\infty) \Rightarrow y \in [0,\infty)$ +1 pt for specifying range of y

$$P(Y \le y) = P\left(\frac{216}{x} \le y\right)$$ + 1pt for correctly setting up probability to find CDF

$$= P\left(x \ge \frac{216}{y}\right)$$

$$= 1 - P\left(x \le \frac{216}{y}\right)$$

$$= 1 - \left(1 - e^{-\frac{216}{10y}}\right)$$ +1 pt for evaluating and substituting CDF of X to get CDF of Y

$$= e^{-\frac{21.6}{y}}$$, which is the CDF

$$\frac{d}{dy} e^{-\frac{21.6}{y}} = \frac{21.6}{y^2} e^{-\frac{21.6}{y}} \quad y \in [0,\infty)$$, which is the pdf

+1 pt for taking derivative and pairing with range to get pdf

Alternative method: change of variables

+1pt $u(x) = \frac{216}{x} \quad v(y) = \frac{216}{y} \quad v'(y) = -\frac{216}{y^2}$

+1pt $f_y(y) = f_x(v(y))\,|v'(y)|$

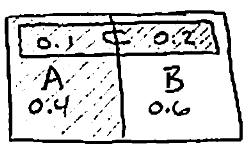
+1pt $= \frac{1}{10} e^{-\frac{v(y)}{10}} |v'(y)|$

+1pt $\left\{ \begin{array}{l} = \frac{1}{10} e^{-\frac{216}{10y}} |v'(y)| \\ = \frac{1}{10} e^{\frac{216}{y}} \left|-\frac{216}{y^2}\right| \end{array} \right.$

+1pt (bounds) $= \frac{21.6}{y^2} e^{-\frac{21.6}{y}} \quad y \in [0,\infty)$

4. Compute the following three probabilities.

(a) Events A and B are a partition. $P(B) = 0.6$, $P(A \cap C) = 0.1$ and $P(B \cap C) = 0.2$. What is $P(A \cup C)$? (4)



$P(A) = 1 - P(B)$ (definition of partition) + 2pt

$= 0.4$

$P(A \cup C) = P(A) + P(C \cap A^c)$

$= P(A) + P(C \cap B)$ (definition of partition) + 2pt

$= 0.4 + 0.2$

$= 0.6$

(b) Let the random variable X have pmf $f_X(x) = \frac{6}{\pi^2 x^2}, x = 1, 2, \ldots$. What is $P(1 \leq X \leq 3)$? (3)

$$P(1 \leq x \leq 3) = \sum_{x=1}^{3} f_x(x)$$

+1 pt for sum
+1 pt for correct bounds
+1 pt for substituting f(x) and evaluating

$$= \sum_{x=1}^{3} \frac{6}{\pi^2 x^2}$$

$$= \frac{6}{\pi^2} + \frac{6}{\pi^2 4} + \frac{6}{\pi^2 9}$$

$$= \frac{49}{\pi^2 6} = 0.827$$

(c) Let the random variable Y have pdf $f_Y(y) = \frac{1}{y^2}$. What is $P(1 \leq Y \leq 3)$? (3)

pdf here implies that y is continuous

$$\therefore P(1 \leq y \leq 3) = \int^3 f_y(y)\,dy$$

+1 pt for integral formula
+1 pt for correct bounds
+1 pt for substituting f(y) and solving

$$= \int_1^3 \frac{1}{y^2}\,dy$$

$$= -\frac{1}{y}\Big|_1^3$$

$$= -\frac{1}{3} + 1$$

$$= \frac{2}{3}$$

Pledge (including time started and finished):

Question	Points	Score
1	8	
2	10	
3	12	
4	10	
Total:	40	