Stat310: Test 1

Answer the questions in the spaces provided on the question sheets. If you run out of room for an answer, continue on the back of the page.
You have **80 minutes** and you may use one double-sided page of notes. You may not use your text book or a computer (calculators are fine). Please pledge on the last page.

Name: Hadley Wickham

1. I call my bank and the recording tells me there are 4 people ahead of me. I know on average that a call takes 3 minutes.

   (a) How long do I expect to wait? Why? (2)

   $4 \times 3 = 12$ minutes – expectation is a linear operator

   1pt =12 1pt Justification

   (b) What is a plausible distribution for the amount of time I wait? Justify your choice. (3)

   Exponential(~~1/3~~ 3)

   1pt 1pt = $\frac{1}{3}$ 1pt Justification

   (c) If I had a way of generating random numbers from a uniform distribution, how could I use them to simulate from that distribution? (2)

   Transform the samples from a random uniform with the inverse cdf of the exponential

   (d) Let N be the number of calls the bank receives in an hour. What's a plausible distribution for N? (3)

   $N \sim \text{Poisson}(60/3 = 20)$

   1pt 1pt 1pt justification "counts"

2. The following probability questions require you to explore some interesting findings about birthdays.

(a) If there are 10 people in a room, what is the probability that any two of them share a birthday? (3)
(Hint: start by figuring out the probability that none of them share a birthday)

7/2

$$\frac{365}{365} \times \frac{364}{365} \times \frac{363}{365} \times \frac{362}{365} \times \frac{361}{365} \overset{(1\ pt)}{=} \frac{365!}{365^{5}\,(365-5)!} = 0.973$$

(1 pt)

P(some one shares) = 1 − 0.973 = 0.027 (1 pt)

=2

$$\frac{365}{365} \times \frac{1}{365} \times \frac{36\cancel{4}}{365} \times \frac{363}{365}$$

(b) If there are n people in room, what is the probability that any two of them share a birthday? (2)

$$1 - \frac{365!}{365^{n}\,(365-n)!} \qquad n < 365$$

if previous answer incorrect, but generalized correctly still give 2 pt.

(c) There are some interesting seasonal patterns in birth date: while unmarried mothers are equally likely to give birth in any month, married mothers are more likely to give birth in spring and less likely to give birth in winter. Assume that only 0.075 of married mothers give birth in winter and about 30% of babies are born to unmarried mothers. Given that a child was born in winter, what's the probability that their mother was married? (5)

100,000

30,000 unmarried | 70,000 married

2,500 born in winter | 5250

$$\frac{5250}{2500 + 5250} = \cancel{0.677}$$

0.41177

1pt writing prob's correctly
1pt identifying we want P(married | winter)
1pt Bayes
1pt total prob
1pt correct answer

3. I've decided to decorate my office by painting randomly sized dots on the wall. For each dot, I randomly choose the radius by drawing it from a exponential distribution with mean 5. Some selected values of the cdf are shown in the table below.

X	$P(X \leq x)$
1	0.18
2	0.33
3	0.45
4	0.55
5	0.63
6	0.70
7	0.75
8	0.80
9	0.83
10	0.86

X	$P(X \leq x)$
11	0.89
12	0.91
13	0.93
14	0.94
15	0.95
16	0.96
17	0.97
18	0.97
19	0.98
20	0.98

(a) What's the probability that the radius of a dot is between 10 and 20cm? (2)

$$P(10 \leq X \leq 20) = P(X \leq 20) - P(X \leq 10)$$ 1pt basic idea

$$= 0.98 - 0.86$$

$$= 0.12$$ 1pt working

(b) What's the approximate probability that the area of a dot lies between 100 and 300 cm^2? (2)

$$P(100 \leq A \leq 300) = P(100 \leq \pi X^2 \leq 300)$$ 1 pt basic idea

$$= P(5.6 \leq X \leq 9.7) = 0.83 - 0.7 = 0.13$$ 1 pt working

or 0.86 or 0.63

(c) What size area do I need to leave blank to guarantee that at least 95% of the dots will fit in? (2)

$$P(X \leq ?) = 0.95 \Rightarrow x = 15$$ 1 pt basic idea

$$A = \pi 15^2 = 707$$ 1 pt working

(d) Work out the pdf for A, the area of each dot (4)

$A = \pi X^2 \qquad u(x) = \pi x^2 \qquad v(x) = \sqrt{\frac{x}{\pi}}$ 1pt

$f(x) = \frac{1}{5} e^{-x/5}$ 1pt

$v'(x) = \frac{1}{2\sqrt{\pi x}}$

$f_A(a) = f_X(v(a))\,|v'(a)|$ 1pt

$= \frac{1}{5} e^{-\sqrt{\frac{a}{\pi}}/5} \frac{1}{2\sqrt{\pi a}} \qquad a > 0$ 1pt

4. Let $f(x) = \frac{1}{2b} e^{-|x|/b}$, $x \in (-\infty, \infty), b > 0$. (Hint: if you get stuck, try sketching a plot of $f(x)$)

(a) Verify that $f(x)$ is a pdf. (5)

1pt for stating both conditions

$f(x) = \underbrace{\frac{1}{2b}}_{+ve} \underbrace{e^{-|x|/b}}_{+ve} \Rightarrow +ve$ 1pt

NTS $\int_{-\infty}^{\infty} f(x)\, dx = 1$

1pt

$\int_{-\infty}^{\infty} f(x)\, dx = \int_{-\infty}^{\infty} \frac{1}{2b} e^{-|x|/b}\, dx = \cancel{2} \int_0^{\infty} \frac{1}{\cancel{2}b} e^{-x/b}\, dx$ 1pt

$= \left[-e^{-x} \right]_0^{\infty} = -\lim_{x \to \infty} e^{-x} - e^{-0} = 1$ 1pt working

(b) Without using calculus, argue that the mean is 0. (2)

The function is symmetric about 0.

The mean is the balance point $\Rightarrow$ mean $= 0$

2 pt for words to that effect or if you did use calc.

(c) Complete the steps below to work out the mgf. (3)

$$M_X(t) = E(e^{tX})$$ definition of mgf

$$= \int_{-\infty}^{\infty} e^{tx} f(x)\, dx$$ by definition of expectation

$$= \int_{-\infty}^{\infty} e^{tx} e^{-|x|/b}\, dx$$ given $f(x)$

1 pt

$$= \int_{-\infty}^{0} e^{tx + x/b}\, dx + \int_{0}^{\infty} e^{tx - x/b}\, dx$$ to remove absolute value sign

$$= \frac{1}{2b}\left[\frac{b}{1-bt} + \frac{b}{bt+1} \right]$$ integration

1 pt

$$= \frac{1}{2b} \frac{b(bt+1) + b(1-bt)}{(1-bt)(bt+1)}$$ common denominator

$$= \frac{1}{\cancel{2b}} \frac{\cancel{b}(\cancel{bt}+1+1-\cancel{bt})}{1+b^2t^2} = \frac{1}{1+b^2t^2}$$ □

1 pt

Pledge:

I used all means at my disposal to ensure these answers are correct

Question	Points	Score
1	10	
2	10	
3	10	
4	10	
Total:	40	