Stat310: Final exam

Answer the questions in the spaces provided on the question sheets. If you run out of room for an answer, continue on the back of the page.
You have **120 minutes** and you may use three double-sided page of notes. You may not use your text book or a computer (calculators are fine). Please pledge on the last page, and note the times you started and ended.

Name: ______________________________

1. X is a random variable with $E(X) = \mu$ and $Var(X) = \sigma^2$. Let $Y = aX + b$. (5)
   What is cor(XY) if $a > 0$? If $a < 0$?

$$\text{cor}(XY) = \frac{\text{cov}(XY)}{\sqrt{\text{var}(X)\,\text{var}(Y)}} \quad (1) \qquad \text{cov}(XY) = E(XY) - E(X)E(Y)$$

$$\text{Var}(X) = \sigma^2 \quad (1)$$

$$\text{Var}(Y) = \text{Var}(aX+b) = a^2\sigma^2$$

$$\begin{aligned}
\text{cov}(XY) &= E(XY) - E(X)E(Y) \\
&= E(aX^2 + bX) - \mu(a\mu + b) \\
&= aE(X^2) + \cancel{b\mu} - a\mu^2 - \cancel{b\mu} \\
&= a(E(X^2) - E(X)^2) = a\,\text{Var}(X) = a\sigma^2 \quad (1)
\end{aligned}$$

$$\text{cor}(XY) = \frac{a\sigma^2}{\sqrt{\sigma^2 a^2 \sigma^2}} = \frac{a\cancel{\sigma^2}}{|a|\cancel{\sigma^2}} = \frac{a}{|a|}$$

key insight is $\sqrt{a^2} = |a|$ (1)

$$\Rightarrow \text{cor}(XY) = \begin{cases} 1 & \text{if } a > 0 \\ -1 & \text{if } a < 0 \end{cases} \quad \square$$

(1)

If mention $Y = aX+b$ is a linear transformation and use that fact to find correct answers give full marks

2. (a) Using the axioms of probability, show that $P(A \cup B) \leq P(A) + P(B)$. (3)

$P(A) = P(A \cap B) + P(A \cap B^c)$ mutually disjoint

$P(B) = P(A \cap B) + P(A^c \cap B)$ (1)

$P(A \cup B) = P(A \cap B^c) + P(A \cap B) + P(A^c \cap B)$ (1) mutually disjoint

$= (P(A) - P(A \cap B)) + P(A \cap B) + (P(B) - P(A \cap B))$

$= P(A) + P(B) - P(A \cap B)$

$P(A \cap B) \geq 0 \qquad (P(A) \geq 0 \; \forall A \subset S)$

Or 2 points if use $P(A) + P(B) \mp P(A \cap B)$ directly

$\Rightarrow \quad P(A \cup B) \geq P(A) + P(B)$ (1)

(b) When does $P(A \cup B) = P(A) + P(B)$? (1)

$P(A \cap B) = 0 \quad \Rightarrow \quad A \cap B = \phi$

(c) If A and B are exhaustive, $P(A) = 0.4$, $P(B) = 0.7$, what is $P(A \cap B)$? (1)

$P(A \cap B) = 0.4 + 0.7 - 1 = 0.1$

3. (a) Using the definition of expectation, show that for any symmetric pdf (e.g. $f(x) = \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}}$) all odd moments (e.g. $E(X)$, $E(X^3)$) are 0. (3)

(1)

$$E(X^n) = \int_{-\infty}^{\infty} x^n f(x)\, dx = \int_{0}^{\infty} x^n f(x)\, dx + \int_{-\infty}^{0} x^n f(x)\, dx$$

$$= \int_{0}^{\infty} x^n f(x)\, dx + \int_{0}^{\infty} (-x)^n f(-x)\, dx$$

(1) n is odd (1) f is symmetric/even

$$= \int_{0}^{\infty} x^n f(x)\, dx - \int_{0}^{\infty} x^n f(x)\, dx$$

$$= 0$$

Or (3) for similarly reasoning that $f(x)g(x)$ is odd and odd function integrates to 0.

(b) Show how to use the mgf to find out the value of the nth moment. (2)

$$M_X^{(n)}(0)$$

differentiate n times (1)

evaluate at $t=0$ (1)

4. The prostate specific antigen (PSA) test is used to detect prostate cancer. A "positive" test is consistent with cancer, and a "negative" with being disease free. The test has a sensitivity (the probability of being positive if the person has cancer) of 86%, and a specificity (the probably of being negative if the person doesnt have cancer) of 33%. If the overall rate of prostate cancer is 5%, what is the probability you have cancer if you have a positive PSA test? Is this a good test? (4)

$P(+ \mid C^+) = 0.86 \qquad P(C^+) = 0.05$ (1)

$P(- \mid C^-) = 0.33$

NTF

$$P(C^+ \mid +) = \frac{P(+ \mid C^+)\,P(C^+)}{P(+)} \text{ (1)} = \frac{P(+ \mid C)\,P(C^+)}{P(+ \mid C^+)P(C^+) + P(+ \mid C^-)P(C^-)}$$

$$= \frac{0.85 \times 0.05}{0.86 \times 0.05 + (1-0.33)} = 0.063 \text{ (1)}$$

This is not a good test because P(C+ | +) is only just higher that P(C+)! (1)

5. Construct a 95% confidence interval for the sample mean if $\bar{x} = 100$, $s^2 = 20$, and $n = 100$. (Hint: $P(Z < 0.025) = 1.96$) (4)

$$\frac{\bar{X} - \mu}{s/\sqrt{n}} \dot{\sim} N(0,1) \quad \text{because } n \text{ is large} \quad (1)$$

$$\Rightarrow \quad P\left(-1.96 \leq \frac{\bar{X} - \mu}{s/\sqrt{n}} \leq 1.96\right) \quad (1)$$

$$= P\left(\bar{X} - 1.96\frac{s}{\sqrt{n}} \leq \mu \leq \bar{X} + 1.96\frac{s}{\sqrt{n}}\right)$$

$$\Rightarrow \quad \mu \in \left(\bar{X} - \frac{1.96}{\sqrt{n}} \cdot s, \ \bar{X} + 1.96\frac{s}{\sqrt{n}}\right) \quad (1)$$

$$\Rightarrow \quad \mu \in (99.12, \ 100.88) \quad (1)$$

6. (a) Let $X \sim \text{Normal}(\mu, \sigma^2)$. Using change of variables, find the pdf of $Z = \frac{X-\mu}{\sigma}$. (5)

$$f(x) = \frac{1}{\sqrt{2\pi}\,\sigma} e^{-\frac{(x-\mu)^2}{2\sigma^2}} \quad (1)$$

$$u(x) = \frac{x-\mu}{\sigma} \quad (1)$$

$$v(x) = \sigma x + \mu$$

$$v'(x) = \sigma \quad (1)$$

$$f_z(z) = f_x(v(z))\,|v'(z)| \quad (1)$$

$$= \frac{1}{\sqrt{2\pi}\,\cancel{\sigma}} e^{-\frac{(\sigma x+\mu-\mu)^2}{2\sigma^2}} \cdot |\cancel{\sigma}|$$

$$\sigma > 0$$

$$= \frac{1}{\sqrt{2\pi}\,\sigma} e^{-x^2/2} \quad (1)$$

(b) What is the distribution of Z? (1)

Standard normal $Z \sim N(0,1)$

7. Let $X_1, X_2, \ldots, X_n$ be a random sample from a Rayleigh distribution. A Rayleigh distribution has one parameter, $\sigma > 0$, mean $\sigma\sqrt{\frac{\pi}{2}}$ and pdf $f(x_i) = \frac{x_i}{\sigma^2}\exp(-\frac{x_i^2}{2\sigma^2})$

(a) Find a maximum likelihood estimator of σ (You don't need to verify it's a maximum). (5)

$$f(x_1, \ldots, x_n) = \prod_{i=1}^{n} f(x_i) = \prod_{i=1}^{n} \frac{x_i}{\sigma^2} e^{-\frac{x_i^2}{2\sigma^2}} \quad (1)$$

(b/c iid)

$$= -2n\sigma \prod x_i \, e^{-\frac{\sum x_i^2}{2\sigma^2}}$$

$$\ell(x_1, \ldots, x_n) = -2n \ln(\sigma) + \ln(\prod x_i) - \frac{\sum x_i^2}{2\sigma^2} \quad (1)$$

$$\frac{\partial \ell(x_1, \ldots, x_n)}{\partial \sigma^2} = \frac{-2n}{\sigma} - \frac{\sum x_i^2}{2} \frac{1}{\sigma^3} = 0 \quad (1)$$

(1) correct algebra

$$\Rightarrow -2n - \frac{\sum x_i^2}{n} \frac{1}{\sigma^2} = 0$$

$$\Rightarrow \hat{\sigma}^2 = \frac{1}{2n} \sum x_i^2 \quad \Rightarrow \quad \hat{\sigma} = \sqrt{\frac{1}{2n} \sum x_i^2} \quad (1)$$

(b) Find a method of moments estimator for σ. (1)

$$E(X) = \sigma\sqrt{\frac{\pi}{2}} \quad \Rightarrow \quad \hat{\sigma} = \bar{x}\sqrt{\frac{2}{\pi}}$$

8. Let $X_i \sim \text{Normal}(\mu_x, \sigma_x^2)$, $i = 1, \ldots, n$ and $Y_j \sim \text{Normal}(\mu_y, \sigma_y^2)$, $j = 1, \ldots, m$, and σ_x^2 and σ_y^2 are known constants (e.g. $\sigma_x^2 = 20$, $\sigma_y^2 = 15$)

(a) Derive the distribution of $\bar{X} - \bar{Y}$. (This is a little more general than what we derived in class) (4)

$$\Rightarrow \quad \bar{X} \sim N\left(\mu_x, \frac{\sigma_x^2}{n}\right) \quad (1)$$

$$\bar{Y} \sim N\left(\mu_y, \frac{\sigma_y^2}{m}\right) \quad (1)$$

$$(1)$$

$$\bar{X} - \bar{Y} \sim N\left(\mu_x - \mu_y, \frac{\sigma_x^2}{n} + \frac{\sigma_y^2}{m}\right)$$

because $E(X-Y) = E(X) - E(Y)$

$Var(X-Y) = Var(X) + Var(Y)$

and differences of normals are normal (1)

(b) Find a pivot for $D = \bar{X} - \bar{Y}$, and give its distribution. (1)

$$\frac{D - \mu_x + \mu_y}{\sqrt{\frac{\sigma_x^2}{n} + \frac{\sigma_y^2}{n}}} \sim N(0,1)$$

Pledge:

Time started: Time ended:

Question	Points	Score
1	5	
2	5	
3	5	
4	4	
5	4	
6	6	
7	6	
8	5	
Total:	40	